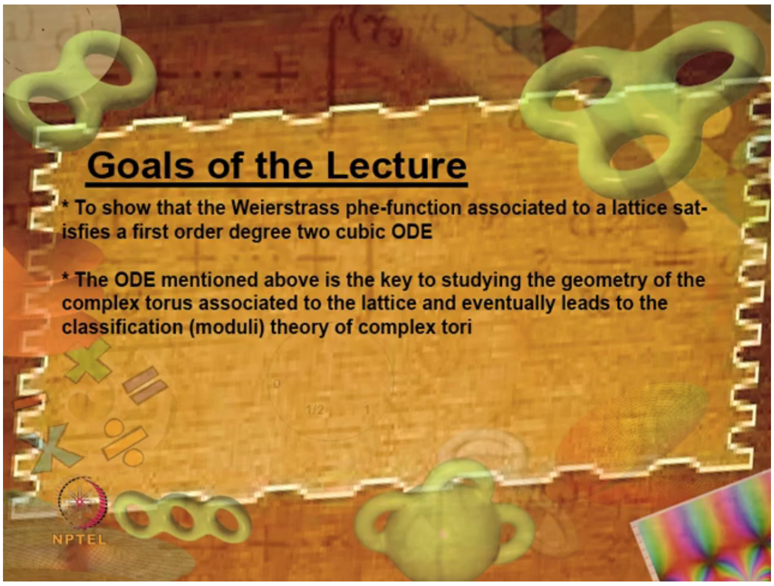
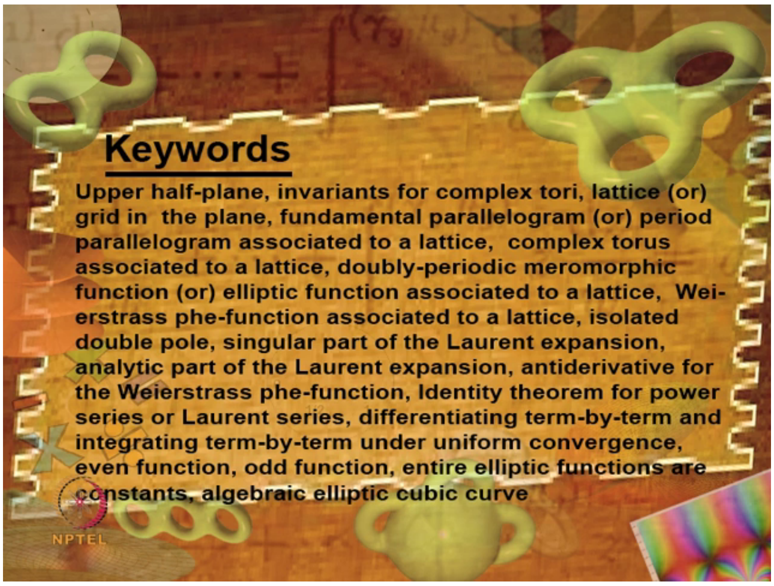
An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-dimensional Tori and Elliptic Curves
Dr. Thiruvalloor Eesanaipaadi Venkata Balaji
Department of Mathematics
Indian Institute of Technology, Madras

Lecture – 31
The First Order Degree Two Cubic Ordinary Differential Equation satisfied by the Weierstrass phe-function

(Refer Slide Time: 00:09)

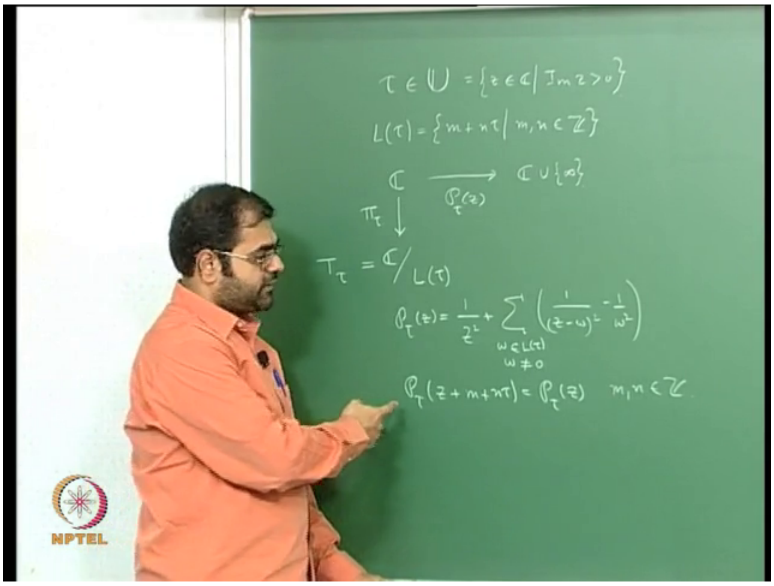


(Refer Slide Time: 00:17)



So, last time we have looked at what the Weierstrass phe-function is which is essentially you know the basic meromorphic function that you can define on a torus. So, let me recall that.

(Refer Slide Time: 00:57)



So, our usual notations, where tau was a complex number in the upper half plane. So, this is a set of all complex numbers z such that imagery part of z is positive. And we associate to tau a torus complex torus, how do we do that we define the lattice L of tau this is the set of all m plus n tau, where m and n are integers; and your torus is just gotten by going modulo this lattice. So, you have the map a natural projection from C to C modulo the lattice, and this is what is called as the torus associated to tau T tau and of course, this map is pi sub tau.

And of course, so this quotient is the set of equivalence classes where the equivalence relation is two complex numbers are e are declared equivalent if their difference is an element of this form. Another way of saying it is that each element of L of tau acts as a translation on the complex numbers and if you go a modulo that action so this is an abelian group which is just isomorphic to z cross z. And this acts on c, and this is precisely the set of orbits. And we know that this is a this is a universal covering with the a deck transformation group equal to the fundamental group of the torus which can be identified with the L of tau yeah.

So, the point is we define the Weierstrass phe-function, so the Weierstrass phe-function was defined. So, phe of z this is phe sub tau of z. So, this was a Weierstrass phe-function that was defined on C union infinity I mean it is defined on c with values in C and infinity the sense that it is a meromorphic function. So, it is a meromorphic function with precisely a double pole at each point of this lattice with sum of residues 0. And we had a series expansion for the Weierstrass phe-function, which was 1 by z squared plus summation over omega in the lattice omega not equal to 0 of 1 by z minus omega the whole squared minus 1 by omega square.

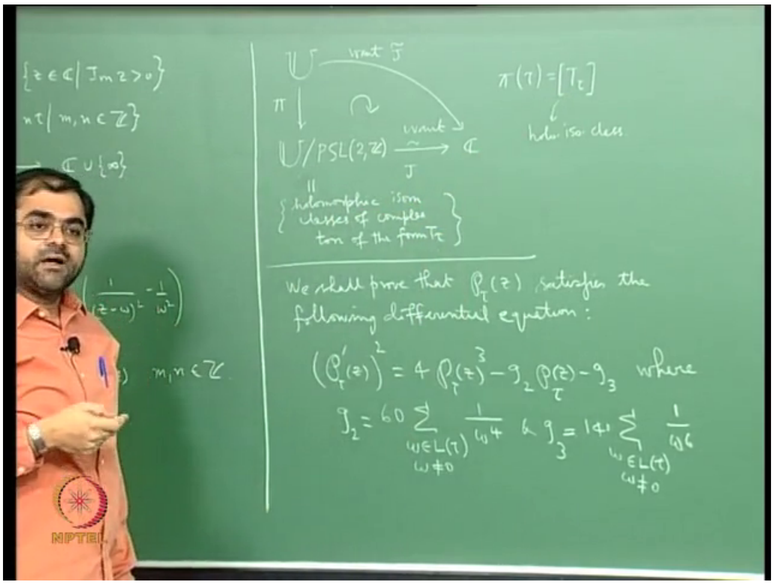
So, we saw that this series converges uniformly and compact subsets outside points of this lattice. And at each point of the lattice of course, this is at each point omega of the lattice including omega equal to 0, this has a pole of order two and there are no other singularities. So, it is a meromorphic function. So, this function is a meromorphic function. And it has been normalized, so that the Laurent expansion at the origin has singular part 1 by z squared, so that was a normalization. And we also know, we also showed that this Weierstrass phe-function is a doubly periodic function. So, it is what is called an elliptic function. It is doubly periodic function, which is meromorphic in the sense that you know if I replace z by any element by translation by any element of this lattice, I get back, I get the same value.

So, one is a period for the Weierstrass phe-function, and tau is also a period for the Weierstrass phe-function that means, phe of is z plus 1 is also phe z from which you can inductively deduce at phe of z plus m for any integer m is also phe of z. And tau is another period and phe which means phe of is z plus tau is phe of z from which you can deduce inductively that phe of is z plus n tau is also phe of z and if you put both together you will get this. So, we proved this last time of course, m and n are of course integers and it is assumed that z is not one of the lattice points because at the lattice points the function is not defined as it is a pole. As z tends to any lattice point you know that this quantity goes to infinity and that is the reason why we write this map as a holomorphic map into cu in infinity, which is the Riemann sphere given the natural complex structure on the Riemann sphere that we have already seen.

So, well, we got the Weierstrass phe-function, because were searching for some functions on the torus. And we realized that we could not have holomorphic functions on the torus with complex values, because they will reduce to constants. Therefore we had allow poles and then we realized that such a function if you have such a function defined here if you composite with pi tau you will get a function on top. So, you will have function on top and that function should be periodic with period one and tau. So, functions on the torus will give you rise to functions on the complex plane which are doubly periodic and conversely a doubly periodic function on the complex plane with periods one and tau will define a function on the torus and then of course, holomorphic functions are not allowed.

So, you are forced to allow, I mean you have your forced allow singularities, I mean by holomorphic functions I mean holomorphic functions are not constant. And then we saw that you the easiest thing we could do is to allow for pole of order two at each lattice point because you cannot have a simple pole because sum of residues will be 0. So, you cannot have a simple pole. So, we had to have a pole either you had to have a pole of order two at each lattice point with sum of residue 0 or you must have allow for two poles with canceling residues, so that the sum of residues is always zero. So, finally, we got this Weierstrass phe-function. So, what is the aim of this lecture. So, you see let me again recall why we are doing all this, so that you know we get the broad picture as I explained in an earlier lecture.

(Refer Slide Time: 09:17)



See our aim was to show that you know or the upper half plane, if you go modulo the upper half plane by the the unimodular group PSL 2, z these are mobius transformations of the form z going to a z plus b by cz plus d with abcd integers. And ad minus bc is actually plus or minus 1. And we see you have already seen how this can be made into a Riemann surface. And our aim is to show that this Riemann surface is actually isomorphic to C, it is actually isomorphic as a Riemann surface to the complex plane, so that means, you would have to find an isomorphism of this Riemann surface with the complex plane.

Of course, you will recall that trying to get this Riemann surface use the fact that we real we realized that even though PSL 2, z has fixed points in its action it was acting we were able to do to prove that it acted properly discontinuously. And the region of discontinuity contained u and therefore, that you can go mod this to get a Riemann surface structure on the quotient, so that this map becomes holomorphic in particular it becomes a maps of map of Riemann surfaces.

And we also noted that over every point of view given a point of view where there is this non trivial stabilizer for this group, we noted that that was a point of ramification. So, this is a ramified cover, but the point is we also next wanted to show that this is isomorphic to C. So, that means, that you have to construct a holomorphic function on this which whose image is c, and which is injective and which is surjective. So, essentially you want to have functions on this. So, you want an isomorphism of this with C, and well I will I will let me call this is for the moment let me call this as j.

So, I want to construct this holomorphic function j, which is an isomorphism of this Riemann surface with c for the complex plane given the natural Riemann surface structure. So, you see, but of course, you know constructing a function like this is would mean in particular that your function like this. It will mean that you have function like this, suppose I call this function as say j tilde, and then this diagram commutes namely this arrow is just this compost to this. And since this is also holomorphic that is also holomorphic, so you are having a holomorphic map from the upper half plain to C. And the fact that this holomorphic map goes down to a map below a will it should tell you that this map j tilde is going to be constant on orbits of PSL 2, z.

So, trying to construct any function here is the same as trying to construct a function on the upper half plane which is constant on PSL 2, z orbits, but we have already seen that the all the elements in PSL 2, z orbit they correspond to exactly to holomorphicly isomorphic complex Tori. So, we have already seen that this is this can be identified with the holomorphic isomorphism classes of complex tori of the form t sub tau and. In fact, you know this map is none other than send a tau in the upper half plane to the holomorphic isomorphism class of the torus defined by the tau.

So, this pi is just the map pi of tau is just you take T tau and then you take this I put the square bracket to say that this is holomorphic isomorphism class. So, if you believe that you want to get the function here, you have to find your function which is constant on holomorphic isomorphism classes of Tori, but usually a function which is constant on holomorphic isomorphism classes in general. If a function is constant or isomorphism classes, it is called an invariant, because it is something that does not change if you change the isomorphism class.

So, the moral of the story is that somehow you are trying to find invariants for complex Tori. And therefore, you see this these invariants for complex Tori should depend on the geometry of the Tori. So, if you believe the philosophy that you know the geometry of a space is controlled by the kind of functions you allow on the face on that space then you will see therefore, that you are led to search for functions on the Tori and that search to that search is what let us to the Weierstrass phe-function.

So, somehow I will have to use the Weierstrass phe-function to get hold of a holomorphic invariant for the Torus that is I have to attach to each Torus a complex number such that this complex number will not change if the torus is changed up to holomorphic isomorphism. And how do I change the torus by holomorphic isomorphism I do that by simply placing tau by another element which is in which is gotten from tau by a unimodular transformation an element of this group. So, this is the story of how we got the Weierstrass phe-function.

And now what I need to do is so somehow I will have to using this phe function somehow, so I have got a function a nice function on each complex torus somehow I have to use it to produce a function on the upper half plane which is going to be invariant under the action of PLS 2, z. So, the key to that is beautiful differential equation satisfied by the Weierstrass phe-function, which is what I would like to prove in this lecture. So, that differential equation is of at most importance also in showing that every holomorphic I mean every complex torus like this actually has an algebraic structure, so that will also come out as a result of that differential equation.

So, let me state what the differential equation is. So, I will just say that if you want j here you want a j tilde above. So, let me draw a line here we shall prove that the Weierstrass phe-function satisfies the following differential equation differential equation. So, what is the differential equation, let me write it down. So, the differential equation is the following it is, so it is derivative of tau derivative of phe tau of z whole squared is equal to 4 times phe tau of z cubed minus g 2 of phe of z phe tau of z minus z 3. So, this is a differential equation.

You can see it is this involves the Weierstrass phe-function and it is derivative. So, it is ordinary differential equation. And you can see that this is this a first order differential equation and you can see this is of degree 2. And what are these g 2 and g 3, this g 2 and g 3 are functions of tau. So, what is this g 2, see g 2 is I will write it down we will compute it and tell and explain what this is, let me see what this is.

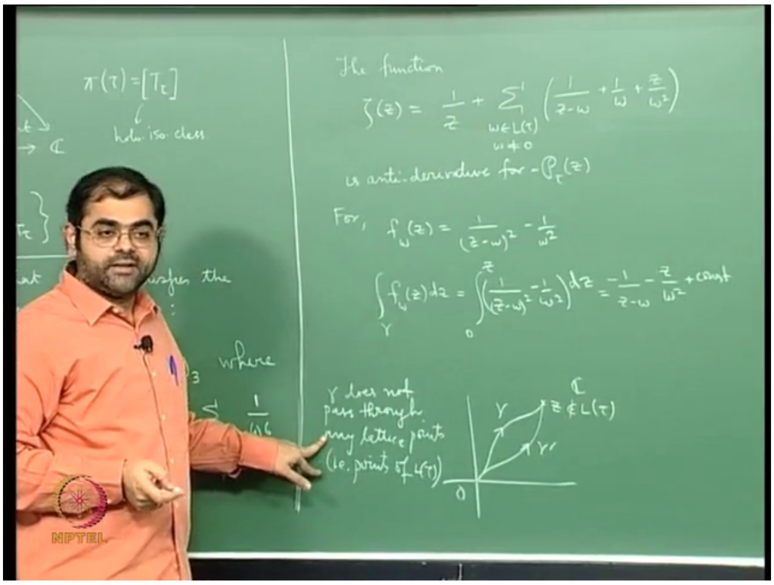
So, this is g, so this is 60 times g 2 I am just referring to my notes, so that I do not make a mistake in writing things down. So, it is 60 times sigma summation over all the lattice points of 1 by omega power 4. So, g 2 is this quantity all right. So, where g 3 is 140 times summation over all the lattice points different from 0 of 1 by omega power 6. So, you see that see g 2 and g 3, they depend on summation on the summations and the summations depend on tau.

So, you must realize the g 2 and g 3 themselves depend on tau. So, g 2 and g 3 are not are just some constants they depend on tau if you think of dependence on tau. And why should you think of dependence on tau because you must always remember that you must think of tau as varying over the upper half plane. So, therefore, you are trying to find an invariant as tau varies you are trying to find any invariant for p tau that is a quantity that does not change if you change t tau or that is if you change tau by an element of PSL 2, z. So, this dependence on tau should all never be forgot. So, this is a differential equation.

Now, let me point out one thing immediately. You can see that this differential equation if you call a phe of z as a variable x and phe prime z as a variable y, then this differential equation becomes a the algebraic cubic equation y squared is equal to 4 x cube minus g 2 x minus g 3 that is an algebraic cubic equation. The equation of a cubic and that is the first evidence to show that a complex torus has an algebraic structure. So, it is actually what is called an algebraic curve and that yeah algebraic curve is called analytical and analytical curve will generally have an equation of this form y squared is equal to 4 x cube minus some constant into x I mean plus some constant with x plus another constant. So, the point is the beautiful thing is that doing so much of analysis, and calculus and getting this differential equation helps you to finally say that your object is algebraic. So, we will see that also.

So, let me try to explain how we get this how we prove this differential equation. So, the key to proving this differential equation is to actually look at it look at the Laurent expansion and the origin. And well the Laurent expansion of the origin for the phe function, we know pretty well that the Laurent expansion at the origin has this as the singular part and this will be the analytic part. Now, the point is that you should write this analytic part in a nice way, so that you can deduce this relationship. And the one way to do that is to first get an anti derivative for the Weierstrass phe-function, so which is what we will do first. So, so let me make the following remark. So, the first remark that I want to make is well you take yeah, so you see the function.

(Refer Slide Time: 23:55)



So, the usual notation is zeta of z, so zeta of z is defined as 1 by z plus summation over elements of the lattice which are not zero and I put 1 by z minus omega plus 1 by omega plus is z by omega square. So, I look at this function and the claim is that this function is an anti derivative for the phe function for in fact for minus p tau of z. So, let me explain why this is this is correct.

So, the first thing is you take the phe function and look at each term here in the in the series that defines the phe function look at each term. So, let us call each term as f sub omega of z. So, f sub omega of z is for me 1 by z minus omega the whole squared minus 1 by omega squared. Now, this is a it is clearly an analytic function except for z equal to omega the particular point z equal to omega, which is the lattice point where it has a pole of order 2.

Now of course, you can integrate this function all right. If you integrate this function over let say any path gamma, which is a path, so this is a complex plane and I choose z to be a point which is not in the lattice. And what I do is that I choose a path gamma from the origin to z and make sure that the path does not pass through any lattice points, gamma does not pass through any lattice points that is you know points of L of tau, you take such a path.

So, you try to suppose I integrate over gamma over that path of suppose I integrate this f omega z dz. Well, this is if I you can see that this is integrating from zero to z of this quantity 1 by z minus omega the whole squared minus 1 by omega squared and dz. Well, I mean if you want to be a little bit more careful then maybe you should call this the dz inside the integration as a different variable because my upper limit is also z, so that should not confuse you. Well if I integrated, you know what I will get is I will get minus 1 by z minus omega minus z by omega square this is what I get plus of course, I will get a constant; this is what I will get. If I naively integrate it, but you have to be careful because this naive integration is correct only when the integral does not depend on the path.

Now, the fact is that the integral does not really depend on the path that is because you see if I replace this path by another path. So, of course, the path has to be directed, so I am going from zero to z suppose I replace it by another path gamma prime. Suppose, I replace it by another path gamma prime and also make sure the gamma prime is also not going to pass through any lattice points, then I claimed integral over gamma is same as the integral over gamma prime that is because you see the integral over this closed path that is gamma prime followed by gamma inverse will actually the integral of that of this function will be by the residue theorem, it will be 2 pi i times sum of residues of this function.

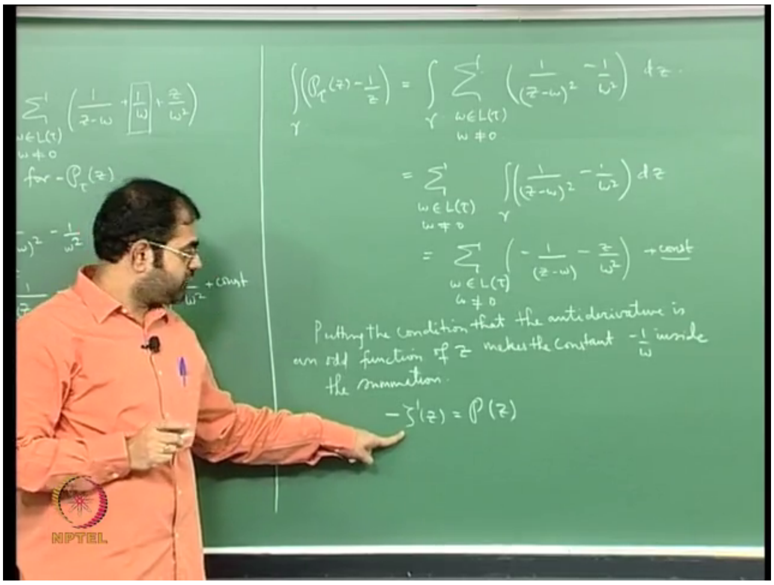
And well there are two possibilities if omega is inside if omega is not inside then anyway there is no problem. Then the function is analytic in this region inside and on the boundary and therefore, the integral does not depend on the path. On the other hand if omega is inside then it will be to pi a sum of residues at that point, but sum of residues is zero because you know this is the reason why we put this in the expression for phe of z is because we wanted a double pole with residue zero. Therefore, this naive integration is correct.

So, you can see that therefore, you know you can see that minus of this which is 1 by z minus omega plus 1 by omega plus is z by omega squared is already these two terms here. Now, the only other thing is to worry about the summation. So, you see that, so I just recall a result that if you have a sequence of functions which are defined on a path, and suppose a sequence of functions converges uniformly to a function. And suppose each of these functions in the sequence is integrable, then if you integrate each of these functions and then take the limit that will be the same as taking the limit and integrating over the path.

So, in other words, so long as a sequence of functions converges uniformly on a path, it will you can interchange limit and integration. And the same argument applied to partial sums of a series of functions which are each of which is continuous will tell you that you can integrate a series of functions term wise. So, long as the series of functions converge is uniformly and we know that the Weierstrass phe-function actually converges uniformly on compact subsets we proved that last time.

Therefore, I can literally integrate this function over such a path and so the integral over the summation can be taken as summation of the term wise integration that is because of uniform convergence; and if I do that this is what I will get, this is what I will get. So, I am sorry I am not integrating this, but I have I am integrating this. So, what I am saying is that, so you see, so let me write that down clearly.

(Refer Slide Time: 32:01)



P tau of z minus 1 by z if I take this quantity and you know integrate it for this path. Then I claim that, so this is going to be integral over summation omega in the lattice omega naught equal to 0 of 1 by z minus omega the whole squared minus 1 by omega squared. And of course, I will get it dz, but you know that this integration and this summation can be interchanged because of uniform convergence on compact subsets. In particular the uniform convergence will also hold on any path which is of course, you know any path is a compact subset because it is say both closed and as well as bounded.

So, you can interchange this summation and integration and get that I can write it like this, and I can write integral over gamma 1 by e z minus omega the whole squared minus 1 by omega squared dz, I can write this. And this is just integral of f sub omega of z dz which turned out to be this. So, finally, what will happen is that you will see that this is going to be this will be summation over omega in L of tau omega not equal to 0 of well minus yeah. So, I will get minus 1 by z minus omega minus z by omega squared plus a constant, this is what I will get. And you see the net effect of all this will be that.

So, you see the only thing that I will have to do is worry about I will have to worry about what concept I am going to add here. So, what I that constant can be uniquely fixed by assuming that the integrated function is an odd function of z, putting the condition that the anti derivative is an odd function of z makes the constant will become 1 by it is going to become, so this constant will be by minus 1 by omega inside the summation. So, if you do that what you will get is that you will get this function you will get this function defined in this way with this constant term 1 by omega coming here. And this constant term has been added, so that this function which is the anti derivative of the Weierstrass phe- is an odd function. You see because if I say change z by minus z this will change by a by a minus sign of course, there is no problem; and this will also change by a minus sign. The only problem is with these two terms.

You see if I change z by minus z then if I want to push the minus outside then I have to change omega to minus omega, therefore you must realize that this constant has to be 1 by omega. I mean if this constant is one by omega then if I change z two minus z I can pull out the minus sign provided I change omega to minus omega; and changing omega to minus omega and the lattice is not going to change this sum. But then changing omega to minus omega allows me to pull a minus out of these two terms and there is already a minus coming out of this term and that term. So, I can pull a minus out throughout therefore, this function becomes an odd function offset. So, the condition that this is an odd function of z fixes this constant inside the summation. And you of course, see that you know if I differentiate this I will get minus of the phe function.

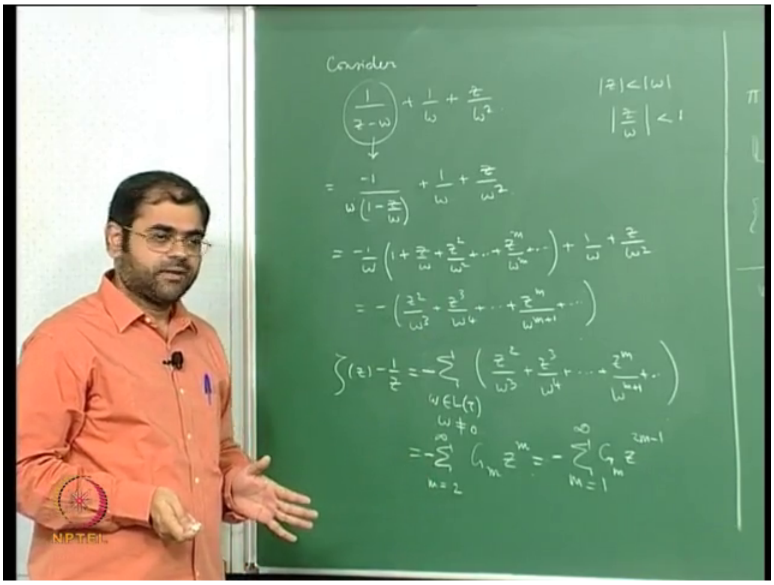
So, let me say that zeta prime of z is well minus of this is the phe of z, this is the phe function. And mainly in principle I should keep remembering the tau because, but for the moment let me drop the tau all right in the subscript. So, I write phe of z instead of phe sub tau of z, please remember that the tau is there in the background. So, I get this. Now, why did I do this what is the advantage of doing this, the advantage of doing this is you see I wanted to prove this differential equation. And as I told you the key to proving this differential equation is by looking at the Laurent expansion at the origin for the phe function. And of course, at the Laurent expansion of the or at the origin for the phe function has this is a singular part and this is the analytic part, but you need to write this since this is an analytic part, you see this is this is going to be a power series in z. And of course, you know we have also normalized the phe function. So, that the constant term is 0.

So, this is going to be an analytic function and you know the Weierstrass phe-function is an even function, therefore, you know if you expand this as a power series in the origin you should get only even powers of z, and you will get some coefficients. And the point is how do you write out those coefficients.

So, the fact is that from this expression directly it is not easy to write it down, but it is easy to write it down f from this expression and from the Laurent expansion of this at the origin which is quite easy to compute. Because you know the Laurent expansion of 1 by z minus omega can be written out using the geometric series. So, the whole point of these argument is you write out the Laurent expansion of zeta of z at the origin using the geometric series because this is the term that will give you the expansion. And then mind you the if I look at the Laurent expansion of this at the origin, this will be the singular part, and this will be the analytic part, but this analytic part is now easy to write because 1 by z minus omega can be expanded using a geometric series,

So, therefore, to put it all together it is easy to write down the Laurent expansion for zeta at the origin n and once you know that and once you know this that the negative derivative of zeta is actually the phe function you can get hold of the Laurent expansion for phe at the origin. So, this is the reason for having introduced this function. So, let us try to get the Laurent expansions.

(Refer Slide Time: 41:00)



So, let me look at only this part of the only that part of the summation, so that is 1 by consider 1 by z minus omega plus 1 by omega plus z by omega squared take this. And what you do is that you assume that mod z say is less than mod omega, you assume this, so that well in other words mod z by omega is less than 1, so that I can apply the geometric series.

So, what happens is that this term you know as usual you can write rewrite it as 1 by omega into 1 minus z by omega, no this should be omega, omega times 1 minus z by omega, you can write it like this. I mean, this is the usual trick that you use to get hold of series expansions using the geometric series. So, if I write it like this, I will have to bring in a minus sign above and then of course, other two terms are 1 by omega plus two by plus z by omega squared.

And of course, when I am certainly z is not equal to omega because mod z by omega is less than 1, and then this is minus 1 by omega n. And if I expand this I will get by geometric series I will get one plus z by omega plus z squared by omega squared and so on, the general term will be z power m by omega power m. I will get this plus 1 by omega plus z by omega square. And you can see that this 1 by omega is going to cancel with this the first term minus 1 by omega and this z by omega squared is going to cancel with minus the second term which is z by omega squared. So, what I am left with is minus of you know z squared by omega cube plus z cube by omega power 4 and so on the general term will be z power m by omega power m plus 1 and so on. So, this is very easy to write down. So, I am just writing down the analytic part of the Laurent expansion of zeta of z at the origin the singular part is of course, is 1 by z.

Now, what you do is that well. So, now, you use this computation to get hold of the Laurent expansion at the origin for zeta, and how does one do it. So, what one does is one take zeta of z minus 1 by z. Now, this difference is actually the summation of terms each of which we have expanded. So, what you can do is that you can write this difference as well, so that I do not make a mistake in signs, let me look at what I have written down. So, it is yeah.

So, I write it as minus. So, this is going to be remember 1 by z is the is a similar part of the Laurent expansion at the origin therefore, this difference is going to be an analytic function. So, it is going to be given by a power series. And I am just going to write it as minus G m z power m, and of course, I will have to put a summation. So, I am going to put the summation over yes. So, I will have to write. So, let me write this clearly. So, minus summation over omega in the lattice omega naught equal to 0 of terms like this correct. So, I will get a series here. So, this is what I will get.

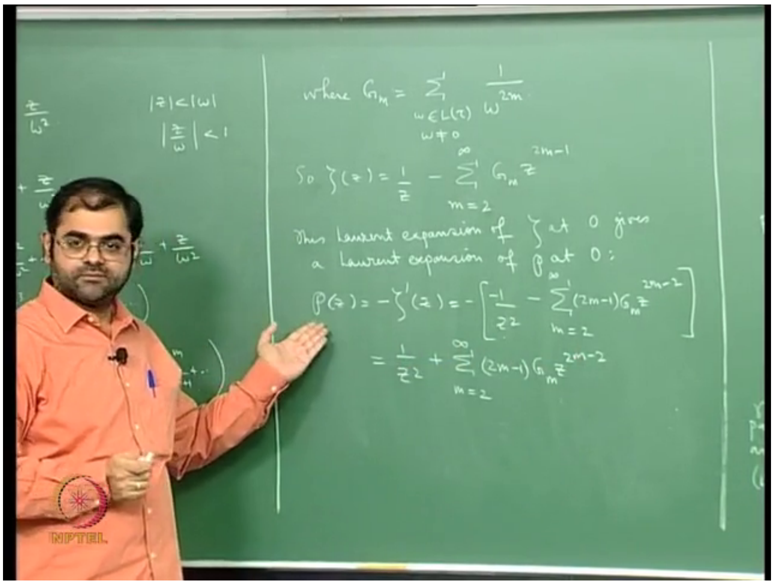
Now, well, I write it as only powers of z. So, if I write as powers of z I write is g sub m. So, I write sigma m equal to 2 to infinity, and I will probably I will put a minus sign outside and put z power m. So, let me write it like this. So, this is going to be you can see that, so I am summing over each of these omegas and for each omega I get z squared z cube and so on. I can sum it the other way. So, what I can do is I can keep z squared fixed and take the summation over 1 by omega cubed and then observe that as a constant which depends on tau essentially. So, I can write it in this form.

But the point is there are two points to be observed, I just want to say that here only odd m will survive, even m will not survive because the quantity on the left side is odd. Because you see if I change z by a minus z, I can pull out a minus sign from here and I can also pull out a minus sign from here so that means, that in when I expand it in the form of a power series I should have only odd powers of z. So, because of that you know I can actually write it as sigma minus of sigma m equal to 1 to infinity G sub m z power 2 m minus 1. Now, I am rewriting it only in terms of odd powers of z namely z, z cube, z power phi and so on that is because the if you sum over, if you keep z squared for example, and sum over all the coefficients that you will become 0, it has to become 0.

Similarly, you do it for any other even power of z, all the coefficients if you sum it has to be equal to 0. The reason is because the left side is an odd function, and therefore, you cannot have even powers series on the right side, this is essentially your use of the identity theorem for power series which says the two power series are equal if and only if all the coefficients are equal.

So, in fact, if two power series convergent power series are equal on set, which has a limit point at which also the power series converges even that is enough to ensure that the power series the analytic function is defined by those power series are everywhere equal. And therefore, the coefficients of power series have to uniquely be related to they have to be the Taylor coefficients for the Taylor expansion of that function. So, therefore, you see here I am switching from writing all the terms to only the odd terms. And what is this G m.

(Refer Slide Time: 49:46)



So, let me write it out G m turns out to be, so where G sub m terms out to be, so it will be summation 1 by 2 power m summation over omega the lattice omega not equal to 0 1 by omega to the 2 m, yes, this is what it would be. You can see because when there is an odd power, you have to sum over omegas of one power more, so it will be in this form. So, if it is z power 2 m minus 1, the term below will be z power omega power 2 m and then you have to sum over all those omegas. So, this is what you will get.

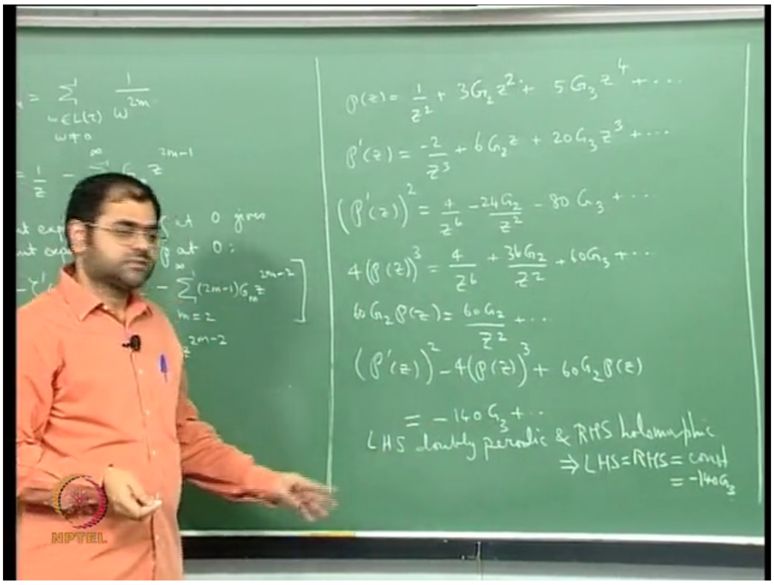
But the point here that we are cleverly using is that the ms for even m they do not survive there are all zero that is because of the oddness of the function on the left side. So, we get this. So, let me write it out. So, you get zeta of z is equal to 1 by z minus sigma m equal to 1 to infinity gm z power 2 m minus 1. I wonder that I put the summation correctly because the first power that I should get is z 3, which means I should start only with two. So, I have to be careful here I have to put 2 here. So, please there is no z term, there is only a z cubed down right. So, here also I should be careful. So, it is m equal to 2 to infinity.

So, now, you see we have a nice Laurent expansion for zeta in a neighborhood of the origin. It is nice because you see the singular part was already known it is 1 by z it was the anal it was the analytic part for which we wanted coefficients in a nice way and that we are able to get. Now, how do you use this to get a Laurent expansion for the phe function, you just make use of the fact that zeta is anti derivative for phe. So, you take the negative derivative of that and because that is going to be again you know uniformly convergent series you can differentiate term by term.

So, as a result what you will get is this Laurent expansion of zeta at origin gives a Laurent expansion of phe at the origin just by term wise differentiation. So, what I will get is phe of z is equal to minus zeta prime of z. So, what I will get is minus of I differentiate it, I will get 1 by minus 1 by z squared and then I will get minus sigma m equal to 2 to infinity I will get 2 m minus 1 G m z power 2 m minus 2. So, you get it very easily. So, this turns out to be 1 by z squared plus sigma m equal to 2 to infinity 2 m minus 1 into gm z power 2 m minus 2.

Now, you see between the original description of the Weierstrass phe-function, and this you see now the analytic part is a very neat. You have all the coefficients of the even powers of z; of course, you will have only even powers of z because this is an even function, but the coefficients were not clear. So, you get the coefficients in a nice way. Now, once you have these coefficients, one can write down a set of inequalities. So, I am just worried about the sign somewhere. So, what one does is that one can differentiate any number of times. So, what one does is just one differentiates, so we are not very far away from establishing the differential equation that we want that zeta that the phe function Weierstrass phe-function satisfies. So, the first thing you do is you differentiate and of course, right only a few relevant terms and it will be clear why these terms are relevant.

(Refer Slide Time: 55:12)



So, let me write down here. So, phe of z is well it is 1 by z squared plus well the first term if I put m equal to 2, I am going to get 3 G 2, I will get 3 G 2 z power z cube if I put m equal to 2 no z squared. Of course, I should it should not get z cube we should get only even powers of z. So, then I will write maybe I will write one more term that the next term is going to be put m equal to 3, I will get 5 G 3 z power you see 4 plus other terms.

And why I put dot dot dot is because I those terms are not going to matter in my calculation. So, this is phe then I calculate the derivative phi prime of z. If we calculate the derivative I end up with well I will get minus 2 by z cube, I can do term by term differentiation because this is a uniformly convergent power series I mean uniformly convergent Laurent series. And I will get I will get 6 G 2 z plus 20 G 3 z cube yeah I stop with that I am not worried about higher order terms.

Then I calculate the square of this p prime z the whole square, I calculate this. And I will get well 4 by z power 6 is one term; then I am going to get I mean the term is time interested in or the I am essentially interested in the singular part plus say the constant term all right. So, I will get 4 by z power 6 and then well I will get my I will get minus 24 G 2 by z square or is it I think it will be yeah z squared that is a product of these two terms multiplied by 2, and then I will also get. So, I will get a minus 80 G 3 and then I will get the higher powers of z and so on. And of course, this term is by you know by taking the product of these two terms in the square and then multiplying by 2, so I get this all right

Then I also calculate 4 phe prime of z, no rather the 4 phe of z cube n this is what I want. And this turns out to be well if you do it by hand it is 4 by, so phe of z is this if I take phe z the whole cube I will get 1 by z power 6, so I will get 4 by z power 6. You can see the reason why I am doing this because you see I can if I subtract then this coefficient goes away it is quite magical rather god given somehow. Then the next term is 36 G 2 by z squared and one more term plus 60 G 3 and so on. And these are of course, higher order terms involving z squared and so on.

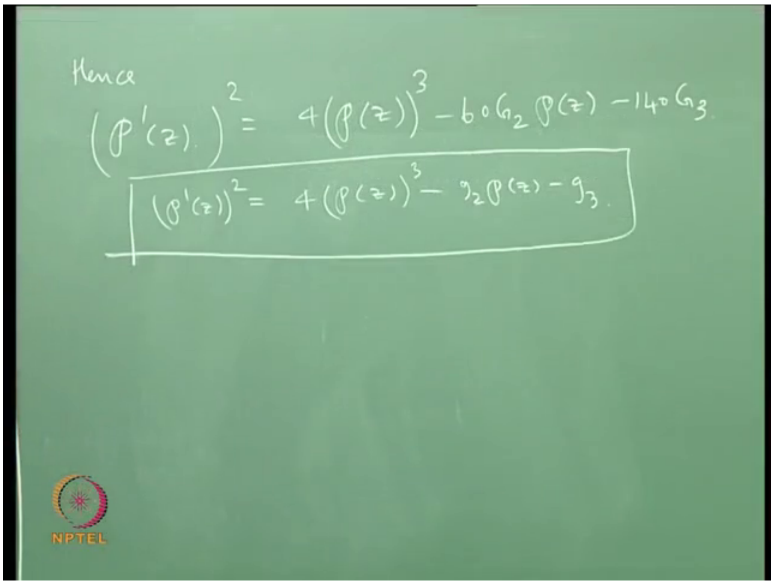
So, now if you calculate and also calculate 60 G 2 times phe of z which is going to give me 60 G 2 by z squared plus. So, I am really not see in the last three equations, in the last three expansions I am really not interested in the power you know terms involving z squared, z power 4 and so on. Now, what I do is you know I just take this minus, this plus, this probably and then I am yeah. So, I take phe prime z the whole squared minus 4 phe of z cube plus 60 G 2 p of z if I calculate this. So, if I calculate this, then what I end up is of course, you see I here this minus this, so 1 by z power 6 is going to go away term is going to go away. And then you see this minus, this is going to be giving minus 60 G 2 by z squared; and if I add it to this plus 60 z 2 by z squared. So, the 1 by z squared term is also going to go away.

So, finally, I will get minus 80 G 3 minus 60 G 3 which is minus 140 G 3. So, what I will get is I will get this is equal to minus 140 G 3 plus some terms which will involve you know z squared and z power 4 and so on. Now, here is a beautiful point, the beautiful point is you see the function on the left side is a doubly periodic function the function on the left side is a doubly periodic function which is a in principle you see phe is meromorphic. So, you expect this side, you do not expect this side to be holomorphic, but on the other hand it is equal to the right side the right side is holomorphic see the right side is minus 140 G 3 plus some you know power series involving even power subset.

So, what you have got is you have got a doubly periodic function the left side is a doubly periodic function because phe is periodic a phe is doubly periodic with periods one and tau and that is equal to a holomorphic function, this is what you are getting. But you see we already know a doubly periodic function which is holomorphic has to be a constant by Liouville's theorem.

So, LHS, RHS, LHS doubly periodic and RHS holomorphic imply that LHS is equal to RHS the left side the left side is equal to the right side is equal to constant this has to be a constant. And what is that constant that constant can be evaluated by taking the right side and substituting z equals 0. If I take the right side and substitute z equal to 0, I will simply get 1 minus 140 G 3, so that constant will be equal to minus 140 G 3.

(Refer Slide Time: 63:36)



So, what is the upshot of this, the upshot of this is that we have the following differential equation which is what we wanted to prove. So, the upshot of this is that the Weierstrass phe-function satisfies the following differential equation. Let me just check the design is I am supposed to get a there is a, so I think yeah minus 140, which is what I wrote as four times p of z the whole cube minus G 2 p of z minus G 3. So, you get this you get this differential equation which is what we wanted to show. Where of course, I have put small g 2 60 times capital G 2 and capital G 2 is summation over all nonzero lattice points of 1 by omega power 4. And g 3 is 140, small g 3 is 140 times capital G 3 and capital G 3 is summation over all nonzero lattice points of 1 by omega power 6. So, you get this nice differential equation, so that was the aim of this lecture.

So, in the forthcoming lectures I will explain how you use this information to get hold of a function on the upper half plane which is going to be invariant under the action of the unimodular group namely a function which will try to give you an invariant for the Tori. So, we will see that in the forthcoming lectures.