An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-dimensional Tori and Elliptic Curves

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Lecture – 30
The Uniqueness Property of the Weierstrass Phe-function associated to a Lattice in the Plane

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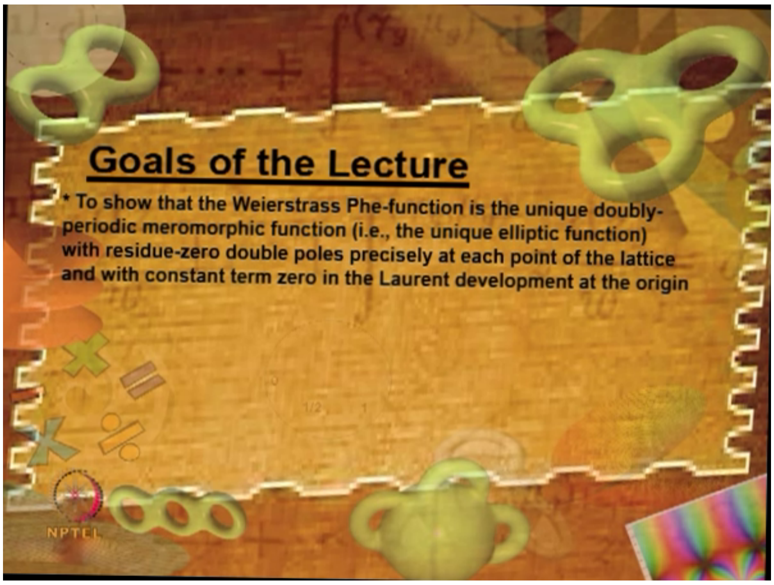

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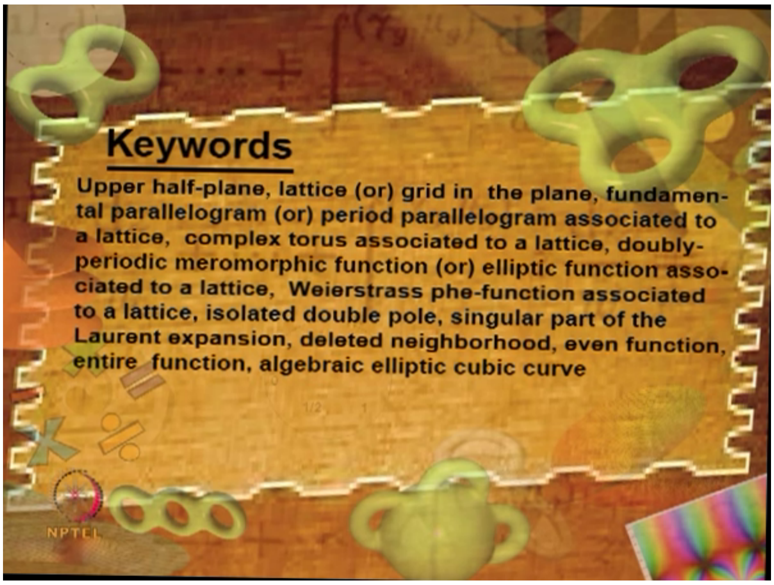

So, in the last lecture, I was trying to convince you that we will have to study meromorphic functions on the torus which are especially the simplest of which are you know a function which has pole at only one point which is a pole of order 2 and so, let me let me go back and write down; I will just recollect what I told you last time, but the purpose of this lecture is to actually show that the that is essentially only one normalized. So, to say type of elliptic function on a torus, which is exactly the west sp function. So, let us again recall.

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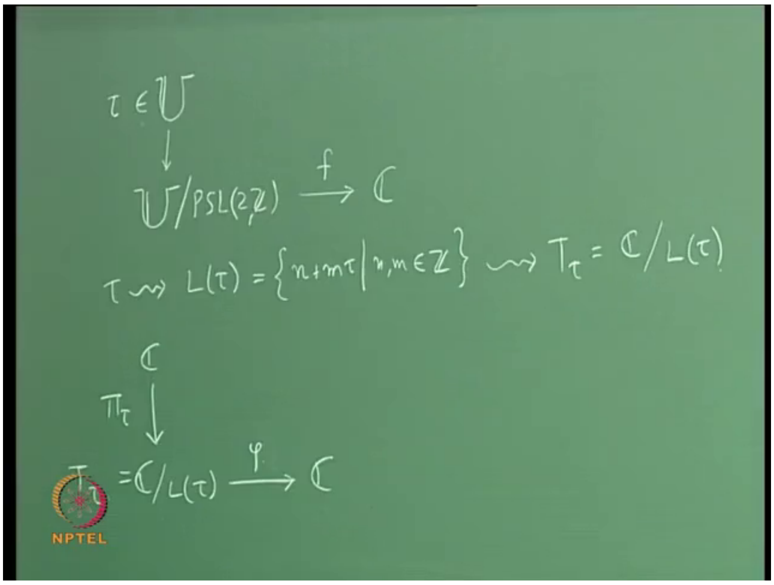

So, very very quickly let me say that you know; I have the upper half plane and there is a quotient of the upper half plane by the unimodular group PSL 2 C and we have seen that this is a Riemann surface and our aim is to show that this is a that this is isomorphic to the complex numbers as Riemann surface ok.

And I told you that in general if you want to even give a holomorphic function from this in to C or for that matter if you want any function even said theoretic function from this into any set it is it is necessarily a function which is constant on orbits and you know the orbits correspond to holomorphic isomorphism classes of complex one dimensional Tori and therefore, you are trying to find a quantity which depends only on the holomorphic isomorphism class of a torus and in it is.

So, it is. So, the point is that looking at a function on this is essentially trying to find the invariants for tori. So, therefore, the point is. So, what are the Tori? I am talking about the Tori I am talking about are well if you take if you take a point tau in the upper half plane then you have the corresponding lattice tau give gives rise to this lattice L tau which is the all the complex numbers of form n plus m tau where n and m are integers and then you can take the complex plane and divide by this lattice to obtain a complex one dimensional torus ok

So, this gives rise to t tau which is C mod L of tau and of course, you know that t tau is a complex one dimensional torus and this the canonical map by tau from C to C model tau

L of tau which is t sub tau this is a nice it is a universal covering map and the fundamental group of the topological space centre underline the torus can be identified with the deck transformation group of this cover which is which is which can be identified with the L tau which is considered as a subgroup of the holomorphic automorphism of C thought of as every element in L tau acting as a translation on c.

So, so essentially I am trying to look at I am trying to produce for each tau I am trying to produce something geometric that depends only on the holomorphic isomorphism class of this torus. So, I am trying to look for an invariant and as I was telling you by the philosophy explain the geometry of space should be controlled by the functions on this space therefore, I am tempted to look at functions on this and I told you that there is no use of looking at you know if I try to think of a holomorphic function. So, do not confuse this f with this f or maybe I will even put a phi 1, all right.

So, if I try to look for holomorphic functions on the torus that is of no use that is because if I compose this map I will get a holomorphic map I will get an entire function because after all this map is automatically holomorphic the way we have constructed it the Riemann surface structure on this is such that this map automatically becomes holomorphic. In fact, you know it is a local locally biholomorphic map because it is a holomorphic universal covering.

So, these are holomorphic map and if this is also holomorphic map the composite is also holomorphic map therefore, you get an entire function on C and then, but then you see this entire function on the other hand if you take the image of the function that is the same as the image of this, but this is a compact set this is compact and this is compact because it is the image of the period parallelogram formed by 1 n tau as the vectors one n tau as coterminous edges.

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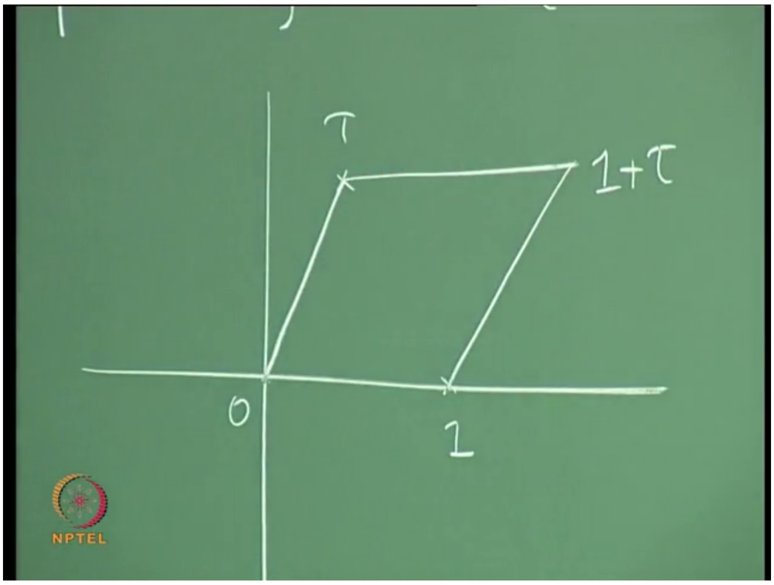

So, the picture is. So, I have; so, here is my. So, here is one and then here is my tau and then I have this fundamental. So, called fundamental parallelogram which corresponds to these 4 vertices which has these 4 vertices and of course, these 4 are the are elements here in this lattice and if I repeat this parallelogram on the whole plane, I will get the vertices will give me all the points in this lattice and the image of this closed parallelogram that is the interior along with the boundary is going to be the image of this function because after all it is periodic.

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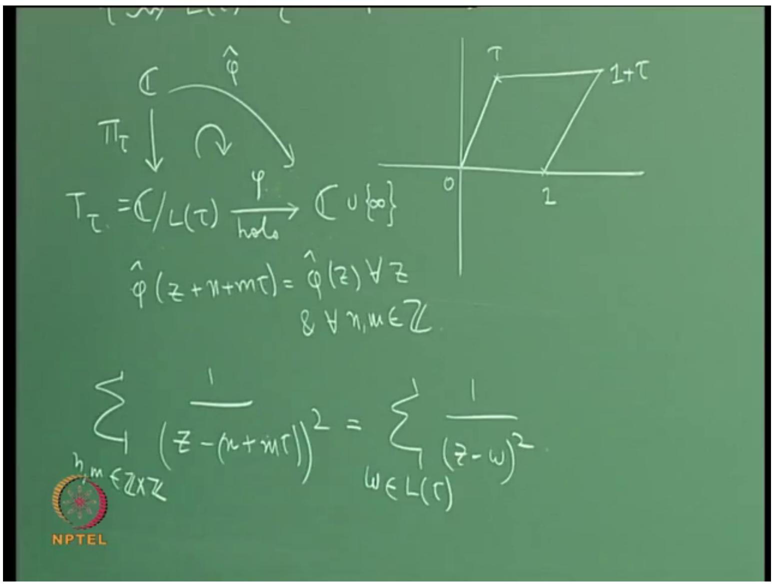

So, the point is that you will if I have a function here, then I get a function there suppose I call it as phi hat then phi hat of course. So, I put this circle or arrow to say that this function is just this forward by this and phi hat of course, has to be periodic with periods one and tau which is the same as which means that phi hat of Z plus n plus m tau should be the same as phi hat of Z for all Z and for all n comma m integers ok.

This is dictated by the fact that I mean this is this is just because phi hat goes down to a function a function above can go down to a function below if and only if it is periodic with respect to this parallel with respect to this lattice that is if the function values are the same if you translate by an element of this lattice. So, such functions. So, you can see for such a function one is a period because phi of EZ plus 1 is phi hat of Z plus 1 is phi hat of Z and phi hat of EZ plus tau is also phi hat of Z. So, 1 and tau are periods. So, these are called W periodic functions and the other name for that are elliptic functions. So, these are elliptic functions, but trying to require that phi is holomorphic is of no use because then the image of phi is the same as the image of under phi hat of this closed parallelogram; that is being clear along with the edges ok.

But then that is a compact set because it is closed and bounded and that and phi being phi hat and phi being continuous the image will be a compact subset of X. So, it will be bounded. So, I will have a bounded holomorphic bounded entire function and by (Refer Time: 09:24) theorem, it has to be a constant therefore, it will not help if I am try to look for holomorphic functions on the complex one dimensional torus I have I am forced to look at their 4 functions with singularities and of course, the main the singularities one can think of are similarities which are poles.

So, I can think of functions which have isolated poles as the only singularities and such functions are called as meromorphic functions and the simplest case is that I can think of a function with a single with a pole at only 1 point and then I have explained to you in the last lecture that if you are going to work with a function which has a pole at only one point then that pole cannot be a simple pole it has to be at least a double pole and then we try to place. So, if the if this function has a pole at the point which is the image of this these lattice mind you all the points in this lattice go to a single point on the complex torus because they are all in the same orbit of this action by this group of translations.

So, if you have a function which has a pole at any one of these points of order 2 that is the same as having a pole of order 2 at the origin all right and. So, you know. So, let me let me rewrite this diagram in the sense that I do not of course, I do not want phi to be holomorphic, I want phi to be meromorphic. So, I will put C union infinity and I will and then I will write holomorphic and this means that you know a holomorphic map into C union infinity is of course, a meromorphic function and the inverse image of infinity are the points that correspond to poles.

So, you see, we wanted a function which had a pole at each point of this lattice and the pole has to be a double pole therefore, you know we could up we were able to guess that roughly well if you want your pole at the point n plus m tau of the lattice then this it should be one simple function that I can write down is of this form n plus m tau whole squared this is the pole of order 2 at n plus m tau and then, but then because of periodicity this should happen if it happens at one point of the lattice, it should happen at every point of the lattice.

So, I will have to take the summation of this over n comma m bearing over integers bearing over integers all right. So, I get this I get this family of I get this. So, double series. So, to say of meromorphic functions each terms is in fact, a rational function and then I have this series all right and the fact is that we are we are; so, this can also be written as you know you can also write it as sigma omega belonging to L of tau of one by EZ minus omega the whole squared you can write it like this I can write it like this and, but this as I will show very soon this is not; this will not give you as it is it will not give you convergence because whenever you write something like this you write a series of meromorphic functions you have to be you have to put some condition.

So, that this series converges at least at the points where there are no poles because where there are no poles you want it to be an analytic function. So, meromorphic function is analytic except for poles all right and the poles have to be isolated in this case anyway the grid points the points of this lattice are of course, isolated.

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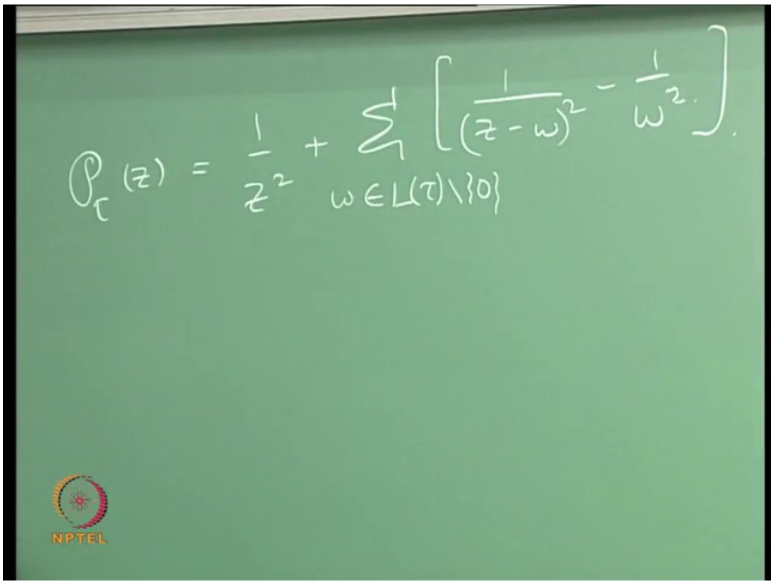

So, to avoid convergence problems what one does is one defines the Weierstrasss phi function in the following way one defines a phi function associated to this; this tau in the upper half plane. So, this associated to this torus as 1 by x squared plus sigma omega a non-zero element of the lattice 1 by 2 Z minus omega the whole squared minus 1 by omega squared. So, this is this is the expression the fact is that you know if you forget this minus 1 by omega squared here, then whatever I have written is exactly what is here they only thing is extra 1 by omega squared is added I have put this one by I have taken out this one by Z squared because you know if I include origin then I cannot put one by 0 squared it does not make sense. So, I had to pull out this Z squared out and then I have taken out this contribution as Z tends to omega this term becomes one by omega squared and I have subtracted that here.

But of course, as Z tends to omega this is going to go to infinity because it is a pole of order two, but the fact is you will see you will see very soon that this is very important to make sure that this converges at all points outside the lattice we will prove that. So, this is the Weierstrasss phi function and this dictates the point is this is essentially the important meromorphic function on this torus and it dictates the geometry of the torus all right it. So, we will see; how it how that is true in the in this and the following lecturers so, but before that I will have to justify that all this is proper I mean well defined and so on and so forth.

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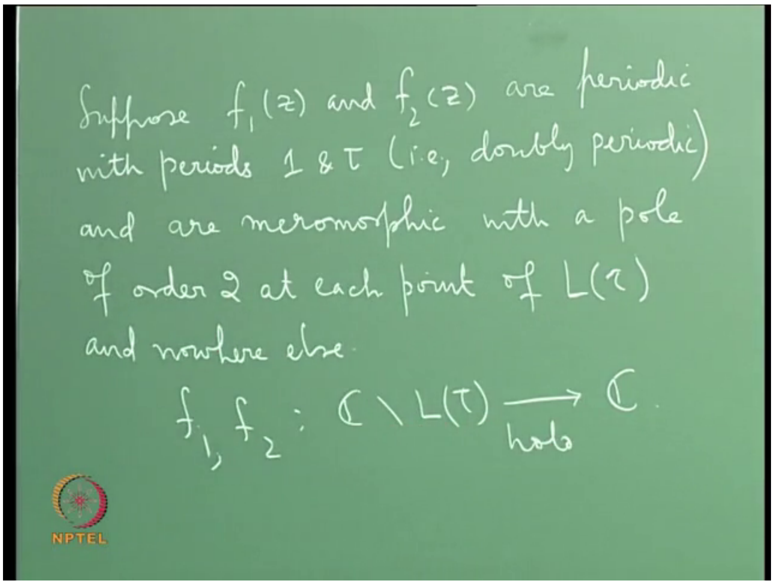

So, let me make a; let me start with a few remarks the first thing is; so, suppose suppose f; let me say f 1 of Z and f 2 of Z suppose I take 2 functions are periodic W periodic or let me say periodic with periods one and tau that is W periodic and let me say that and are and are meromorphic with a pole of order 2 at each point of the lattice and nowhere else ok.

So, you see, I am looking at 2 function, I am looking at 2 metamorphic functions on the complex plane. So, these so; that means, you know f 1 f one comma f 2 are defined from C minus the lattice to C and they are holomorphic, but for the points on the lattice and at each point of the lattice they have a pole of order 2 now. So, so you see what I want to say is that f 1 such functions can be normalized in a certain way.

So, that you finally, end up getting only one unique function and that unique function has to be the Weierstrasss phi function that is called the Weierstrasss phi function by uniqueness and that function is this one has a formula like this. So, what is that normalization? So, the normalization is well we will let me thing for a movement. So, you see f in particular you know if you the normalization is defined by looking at the Laurent expansion or the function in a neighborhood in a deleted neighborhood of the origin see the origin is of course, a point of the lattice and the function has a pole of order 2 at the origin all right and if you write out the Laurent expansion at the origin you see; it will be in the form in the following form.

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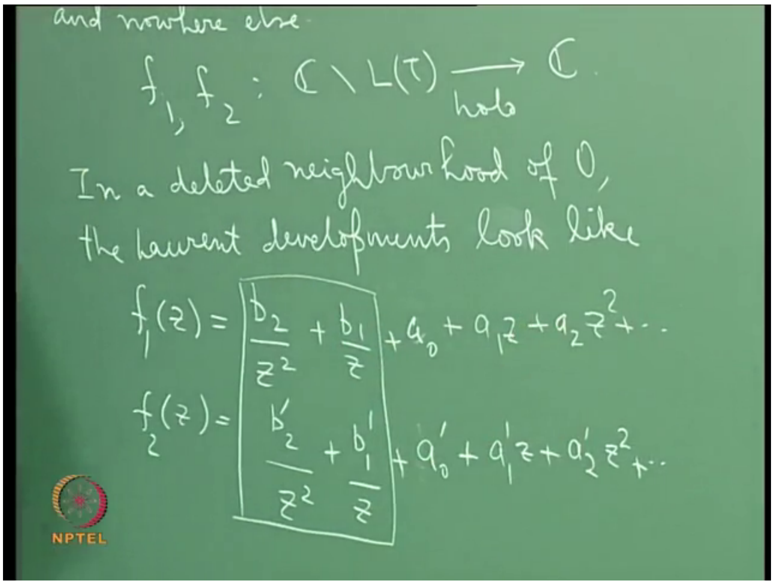

So, let me try that out in a day in a deleted neighborhood of 0 the Laurent; the Laurent developments the Laurent developments are look or let me say in they are; they look like look like f 1 of Z, it is going to be you see it is a pole of order 2. So, what I will get is I will get b 2 by Z squared Z equal to zeroes of pole of order 2. So, I will get b 2 by Z squared plus b 1 by Z plus b naught plus let me say a naught plus a 1 Z plus a 2 Z squared and. So, on and if I take f 2 well I will have to use. So, . So, I will put primes I will say b 2 prime by Z squared plus b 1 prime by Z plus a naught prime plus a 1 prime Z plus a 2 prime Z squared and so on.

The Laurent development is going to look like this that is because Z equal to 0 is a, this is a Laurent development about centered at the origin and the origin is not included. So, it is a punctured neighborhood of the origin and this is the Laurent series development for these 2 functions and the point is that these 2. So, this part this corresponds to the so called singular part of the Laurent development the rest of it is an analytic function the rest of it is an analytic function. So, now, the normalization condition that I put is I will set the coefficient of one by Z squared to be one that is a normalization condition I put.

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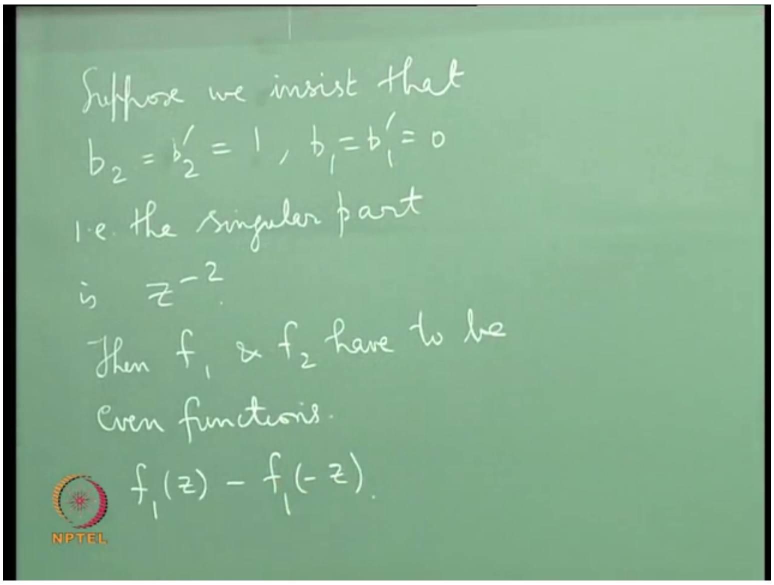

So, suppose we put we insist that b; b 2 is equal to b 2 prime is equal to one suppose if we insist this. So, this is the first normalization condition that is you take the elliptic function all right and look at the Laurent development at the origin and insist that the singular part contains only Z to the minus 2 with a coefficient one and the in particular it means I am setting b 1 and b 1 prime to be 0.

So, let me write that b 1 is equal to b 1 prime is equal to 0 that is in other words, the condition is the singular part a singular part is Z to the minus 2 that is a singular part in general, the singular part will be some it will be some constant into Z to the minus 2 plus some constant Z to the minus 1 the normalization that I do is that I say the singular part has to be Z to the minus 2. Now the claim is that once you do this then any such function automatically becomes an even function. So, that is a first observation all right. So, then f 1 and f 2 have to be even functions f 1 and f 2 have to be even functions see for why see for example, see you look at you look at f 1 of Z minus f 1 of minus Z; look at this function look at the function f 1 Z minus f 1 of minus Z.

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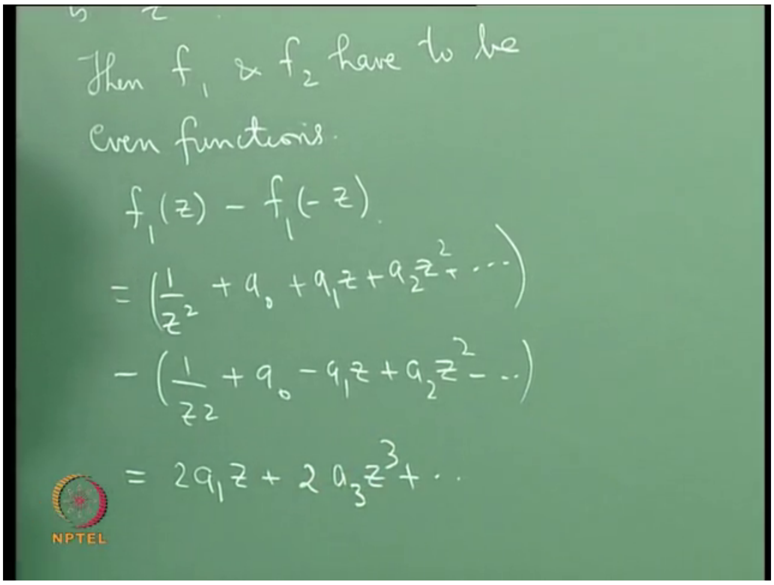

Then you see, you will see that if I explained it what I will get is I will get you see I will get a 1 a 1. So, let me write it down.

So, f 1 Z is going to be well 1 by Z squared plus a naught plus a 1 Z plus a 2 Z squared plus so on minus 12 f 1 of minus Z is going to be 1 by Z square plus a naught minus a 1 Z plus a 2 Z squared minus and so on. So, what I am going to get is I am going to get the point is you see the in a in the neighborhood of the origin the only pole that I have that is removed. So, what happens is I simply get I simply get 2 a 1 2 times a 1 Z plus 2 times a 3 EZs cube and so on and what is this notice that this is a this is an analytic function that the difference is an entire function it is an analytic function and see whatever I have done. So, I am looking at the origin I am looking only at the origin that the if you look at the difference the difference in at a entire function, but on the other hand notice that both of them still have the same periods one and tau.

If I change EZ to EZ plus n plus m tau and here also if I change minus EZ to minus EZ plus n plus m tau because f 1 has n plus m tau as periods one and tau as periods it is still W periodic. So, what is this computation tells you is it you are having a W periodic function which is on a which is which is entire I mean which is which is which has no poles. So, see the point is that this difference is W periodic function it has no pole at the origin therefore, it has no pole at any other lattice point the difference function has no pole at any other lattice point because after all the values of the function are the same if you displace by an element of the lattice translate by an element of the lattice. So, therefore, this difference becomes an entire function all right which is W periodic and it has to be a constant again by (Refer Time: 26:07) theorem.

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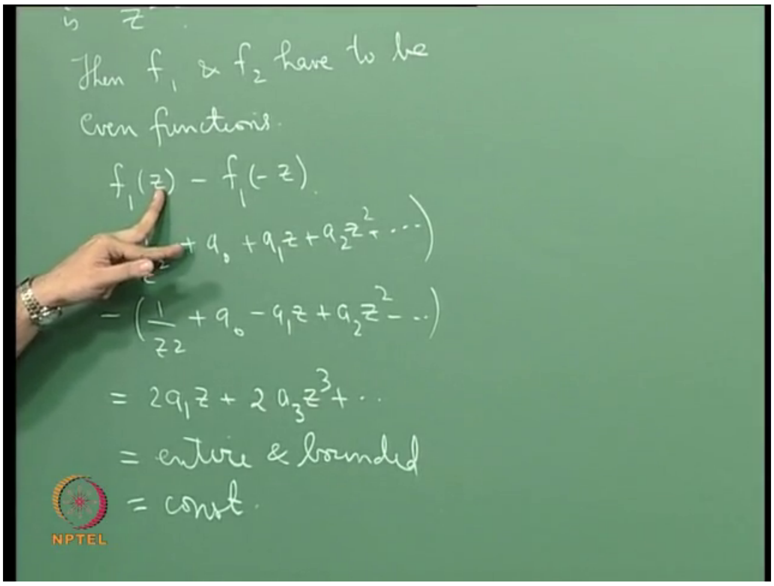

Therefore, this is equal to this; this is entire and bounded; the boundedness comes from the W periodic nature.

So, this will be equal to a constant. So, this difference has to be a constant and the claim is that that constant is actually 0 that constant is actually 0 because if you want you know I can put half put Z equal to half or put Z equal to tau by 2 put something either half or tau by 2.

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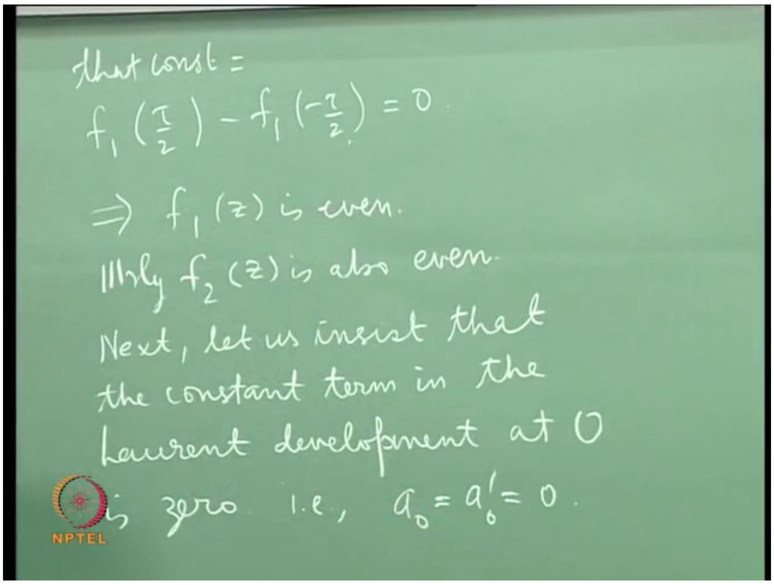

So, you know I will get f 1 of let us say tau by 2 minus f 1 of minus tau by 2 that constant is also going to be equal to this and this is going to 0 this going to be 0 because the difference between these 2 Z values is tau and taw is a period and the function does not change its value if you change the argument by translate the argument by a period. So, the constant is 0 as a result what this argument tells you is that f 1 of Z is even f 1 of Z is even.

So, the point is the moment you put the condition that the singular part is Z to the minus 2 the function becomes even the function becomes even. So, now, I am going to put one more condition for the normalization and that that condition is going to be that when you write out this Laurent expansion I am going to insist that the constant term is 0 I am going to insist the second condition I am going to put is that the constant term is 0 suppose. So, let me write this here similarly f 2 of Z is also even for the same reasons now next let us insist that let us insist that the constant term in the Laurent development in the Laurent development at the origin is 0, let us put this extra condition. So, that means. So, in this discussion I am saying put a naught equal to 0 a naught prime equal to 0 ok.

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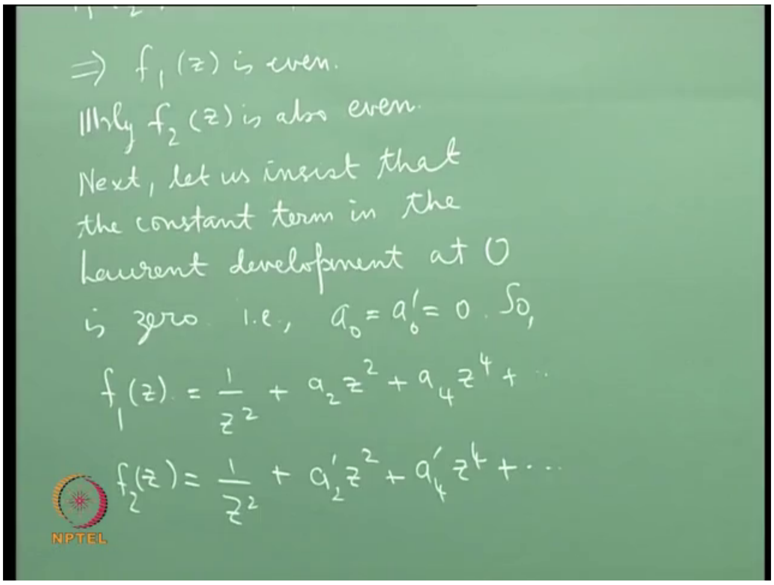

So, you see what will happen is that f 1 of z will now have this expansion, it will be one by Z squared plus a 2 8 squared plus a 4 Z power 4 and so on mind you, it is an even function. So, you have the expansion has to be like this and f 2 of Z will be again one by Z squared plus a 2 prime Z squared plus a 4 prime Z power 4 and so on. So, this how the Laurent expansions are going to look like now you see again if I subtract the 2 again if I subtract the 2, I will get an entire function which is W periodic and therefore, it will be a constant and therefore, these 2 have to be equal.

So, the upshot of the story is you take a W periodic function you that is an elliptic function with exactly a pole of order 2 at each point of the lattice you put these 2 normalization conditions namely in the Laurent expansion at the origin the singular part is 1 by Z squared and the second condition is that the constant term is 0, then there is only 1 function and the claim is that one function is the Weierstrasss phi function and the function is given by this formula that is the claim.

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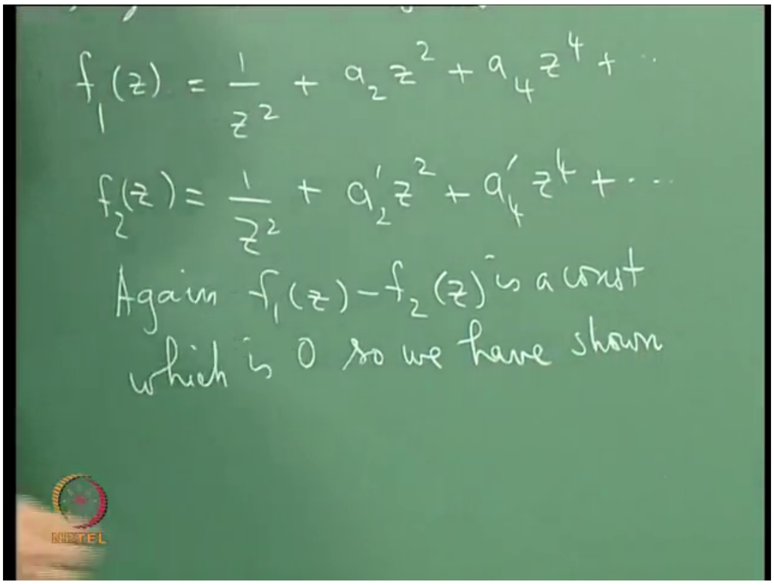

So, let me write that down again, f 1 of Z minus f 2 of Z is a constant which is 0 which is 0. So, so we have we have we have shown. So, what we have shown is the following. So, I can write it here. So, maybe I will draw some lines here. So, that. So, what we have shown is the following theorem.

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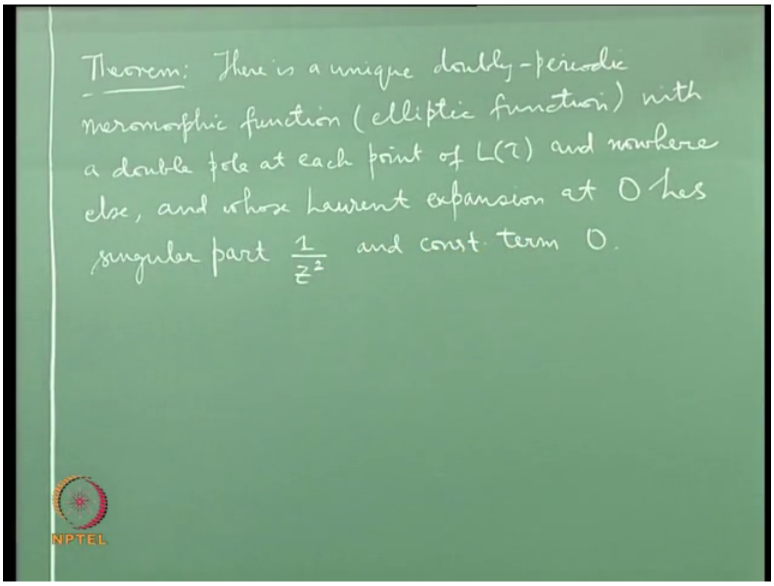

There is a unique W periodic meromorphic function elliptic function with a double pole at each point of L of tau and nowhere else and whose Laurent expansion at the origin has singular part 1 by Z squared and constant term zero. So, what I proved is that there is essentially only one I mean I had given these normalization exactly one W periodic meromorphic function with pole of order 2 at each lattice point and with the extra normalization condition that the Laurent expansion at 0 that is at the singular part is one by Z squared and the constant term is 0. So, this is what this is what the discussion shows.

Now, what I am going to do is I am going to show tell you I am going to call that function as the Weierstrasss phi function that is it that unique function I am going to call it the Weierstrasss phi function associated with the torus. So, let me write that we call this unique function phi taus of Z and the Weierstrasss phi function associated to t tau o[r] or to L tau or 2 tau in the upper half plane. So, this is the; we call this unique function as a Weierstrasss phi function.

So, you must all be wandering that I have written this thing here. So, that is going to be the next thing that we are going to show I am going to show that the right side has all these properties I am going to show that the right side is a meromorphic function it is W periodic of course, the Laurent expansion at the origin the singular part is exactly one by Z squared and that constant term is 0, therefore, this has to be equal to that is what I am going to do next ok.

So, and that will tell you that this is indeed the expression for the Weierstrasss phi function.

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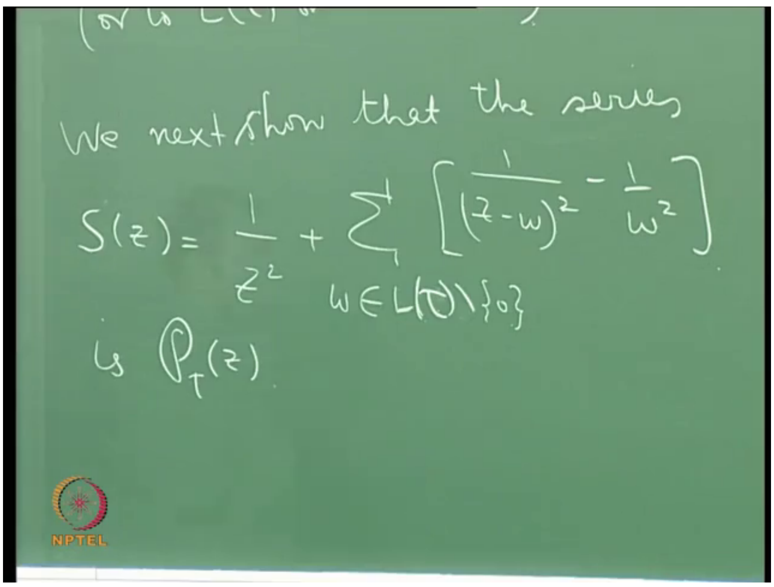

So, let us do that we next show that the series. So, let me let me give it a name phi of Z let me call it as phi of Z let me call it phi of z, but I do not want you to confuse that phi with phi hat. So, let me call this something let me call it as f of Z. So, suppose I call this or maybe even S of Z if it helps. So, S of Z is that series 1 by Z squared plus sigma omega in the lattice omega not equal to 0 lattice omega not equal to 0 of this expression which is 1 by EZ minus omega the whole squared minus 1 by omega squared we next how that the series this series has all these properties and. So, has to be equal to the Weierstrasss phi function is phi is a Weierstrasss phi function we show this ok.

So, let me do that. So, so the first thing that I need to worry about is the convergence the first thing that I need to worry about is the convergence.

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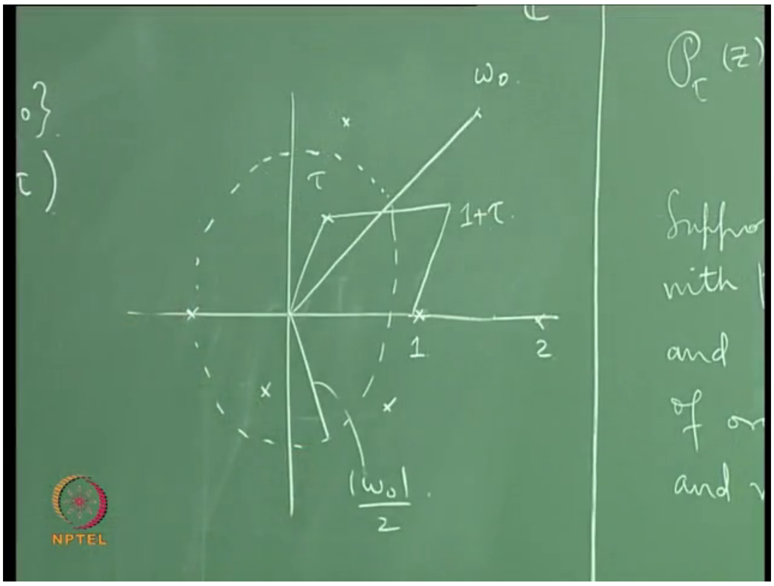

So, you see let me let me draw a diagram. So, picturing the lattice; so, I have. So, this is my tau this is one let me put it here one point. So, this is my. So, and this is one plus tau. So, I have this I have this lattice; this lattice of points repeating themselves.

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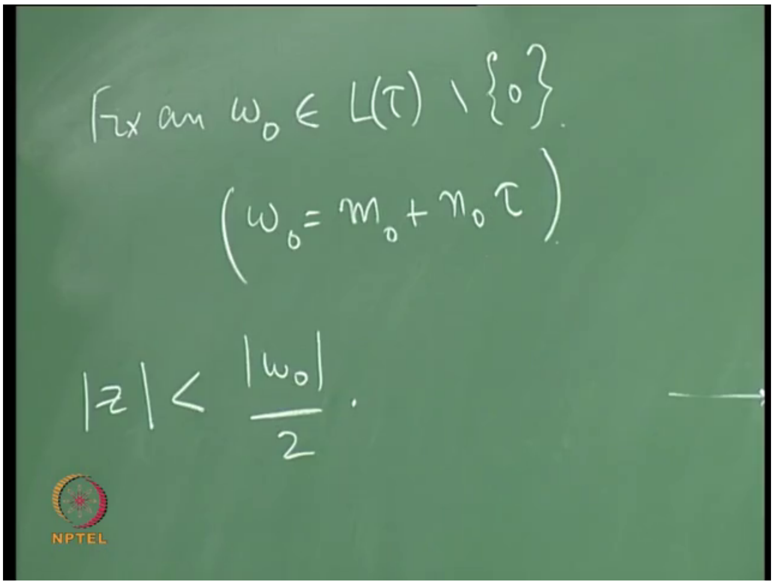

So, I will have 2 and so on and so forth and so, you see fix and omega not in the lattice you fix and omega are not I the lattice which is not which is non zero. So, you see this omega naught is going to be therefore, of the form m naught plus n naught tau where both m naught and n naught are not simultaneously 0 and let us look at the condition mod Z lesser than mod of omega naught by 2 I am going to look at this condition, I am going to look at this open disk.

So, you see well omega naught is for example, some point of the lattice and you know I am, then this is the vector that represents omega naught and the length of this vector is mod omega naught and I am looking at. So, mod omega naught by 2 is going to be something here and therefore, I am looking at this disk. So, this is the radius of; this is mod omega naught by 2 and I am looking at all the all the complex numbers Z here and of course, and I am trying to estimate each term of that series I am. In fact, in particular I would like to estimate the term of this series that corresponds to omega equal to omega naught right. So, let us do that see what is the term inside.

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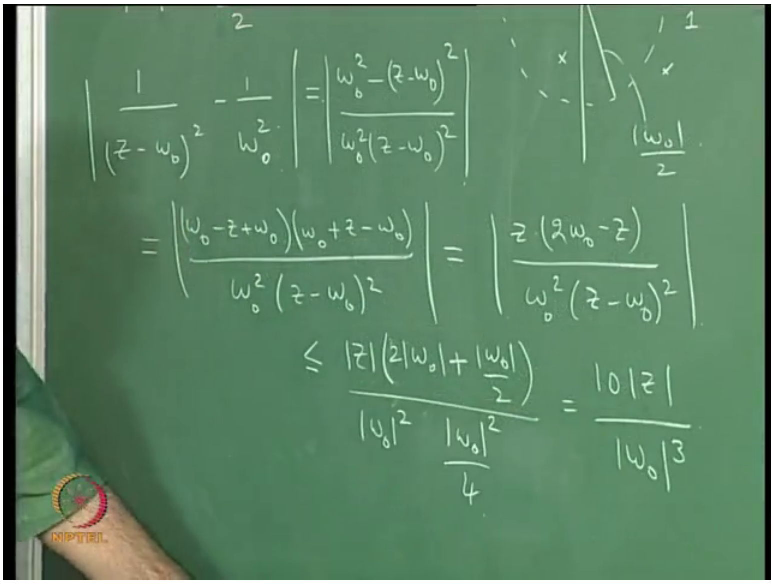

It is 1 by Z minus omega naught the whole square minus 1 by omega naught omega naught squared this what I have and if I take the mod of this what do I get I get well it will be omega naught squared minus Z minus omega naught squared by omega naught squared into Z minus omega naught whole squared mod and this will be 12 this is well two. So, it is omega naught minus Z plus omega naught into this omega naught minus plus Z minus omega naught divided by omega naught squared into Z minus omega naught squared and that turns out to be well I will get Z into 2 omega naught minus Z by omega naught squared into Z minus omega naught squared.

And if I estimate this see I will get this is less than or equal to you see of course, I can replace this Z by mod omega naught by 2 in the numerator, but I do not do it purposely. So, I will get I will keep this mod Z as it is you will see why I am I why I am not doing it the reason for that. So, I am keeping this mod Z as it is all right and this 2 omega naught minus Z mod can be written by triangle inequality to be less than or equal to you know well 2 times mod omega naught plus mod omega naught by 2, I can write this I am just using mod of a plus b is less than or equal to mod a plus mod b and I am using mod Z is less than mod omega naught by 2 and of course, this omega naught squared in the denominator I am I am not going to worry about it and this one this mod Z minus omega naught is by triangle inequality it is greater than equal to modulus of mod Z minus mod omega naught and that will give me. So, the reciprocal of that will be less than or equal to the reciprocal of modulus of mod Z minus mod omega naught this is what I will get all right.

And yeah; so, I just wanted to say that. So, I just wanted to say that this can be. So, yeah this can be replaced by mod omega naught squared by 4 again by using finding inequality all right. So, yeah; so the upshot of all this is I will get well I will get 5 10 10 mod Z by mod omega naught cube. So, I need this you see if I had replaced the mod Z by again by mod omega naught by 2 I will get only a mod omega naught squared and mind you your lattices in 2 dimensions it is not good to have one by mod omega naught mod omega squared and trying to sum it.

So, now what I want you notice is that if mod Z; see if mod Z if see. So, what I am going t; let me say what I am going to write down if you take any points of the lattice which are outside; I am in which are outside the circle centered at the origin and radius mod omega naught, then it is also true that mod Z is also lesser than mod omega by 2 ok and the all these calculation will all the these estimation will be will hold with omega naught replaced by omega.

The point is it will hold for all omega that is out that are outside this that are outside this circle points of the lattice that are outside this circle and what is left out are only finitely many points which are inside the inside the circle. So, the convergence is essentially a question of whether for all points outside the series converges and what we will show is that the series is numerically dominated by some multiple of sigma 1 by n squared which you know it converges. So, and that is where we will use this you will need this omega naught for the cube.

So, let me explain that. So, and of course, in all these calculations; one is of course, assuming that when you sum it up you are assuming that Z is not going to be any one of the; of course, not going to be any one of the lattice points because if Z is any of that omegas in consideration then this term does not make sense because it becomes 0.

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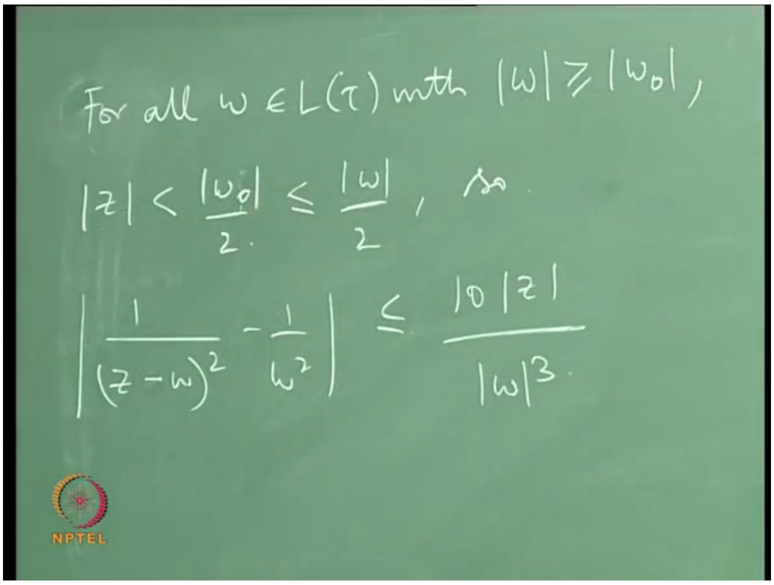

So, you see for all omega in L off tau with omega mod omega greater than or equal to mod omega naught ok.

You will similarly get that mod Z is going to be certainly less than mod omega by 2 this has to happen because mod Z is going to be is less than or equal to mod is less than mod omega 2 and that is less than equal to mod omega by 2. So, the same kind of calculation will tell you that 1 by Z minus omega; the whole squared minus 1 by omega squared this is this is this is bounded by ten mod Z by mod omega k all right. So, therefore, you see if I look at this series.

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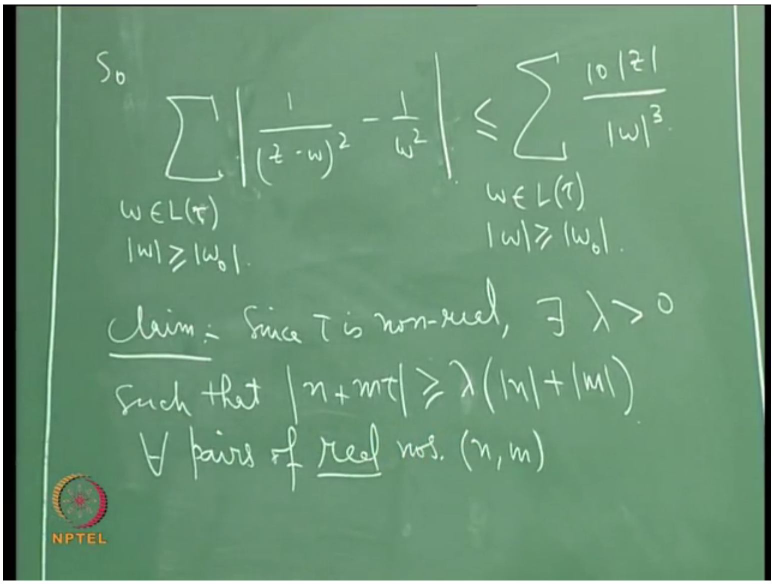

I mean tail end of the series you take omega in L of tau you take all omega in L of tau with mod omega greater than or equal to mod omega naught, then this series one by Z minus omega the whole squared minus 1 by omega squared if I look at this series this is just the that series except for finitely many terms.

So, that converges if and only if this converges all right. So, and if I look at the absolute series if I look at the absolute series what I will get is I will I will get that this is it is going to be by this estimate omega belonging to L of tau summation over mod omega greater than to mod omega naught and I will get ten mod Z by mod omega the whole cube this is what I get now what I am going to say is that this is actually going to converge and for that there is there is little bit geometric in insight that is needed.

So, here is a claim since tau is non real, since tau is non real there exists a lambda greater than 0 for a real lambda greater than 0 such that such that modulus of n plus m tau is always greater than lambda times greater than or equal to lambda times mod n plus mod m. So, you see this is something that you can immediately think of geometrically, but it is an exercise to verify that this holds roughly you know if I draw a diagram at least you should be able to see why this is a result that can be expected because you see and and of course, here n and m need not necessarily be integers n and m could be any pairs of real numbers ok.

So, let me write that also for all pairs of real numbers n command. So, I am saying real numbers. So, in particularly, it will also hold for pairs of integers. So, what does this mean see if you if I draw this if a draw this lattice.

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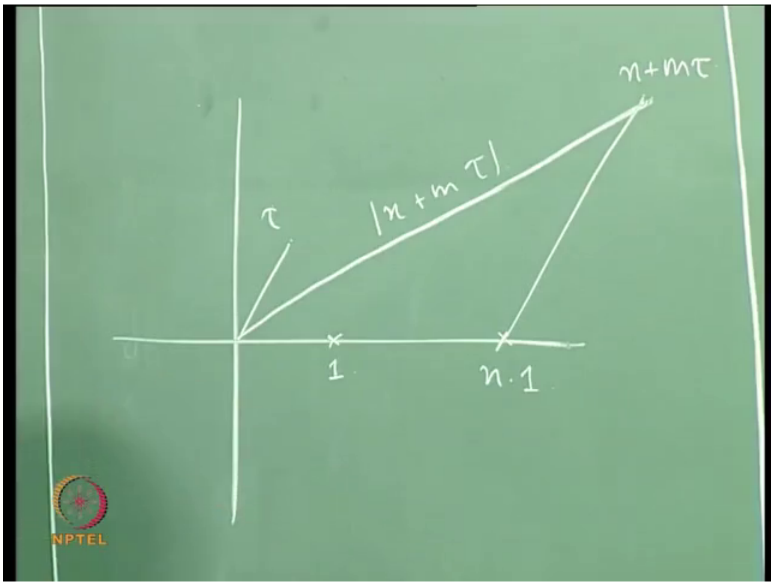

So if I draw this lattice. So, you see I have see let us assume that say n and m are positive. So, you see. So, this will be this point will be n times one and then I will have if this is a direction of if this is tau then I will have some something like this, this will be n plus m tau and you know if I and this is one of course, somewhere here I am assuming n is greater than one in this diagram. So, if I join this you see this length is modulus of n plus m tau and you see if I take modulus of n plus m n plus m tau and if I divide it by mod n plus mod m I am actually dividing by you know this length and by the factor of this length by tau and you can see that the ratio; it cannot go to it has to be bounded below mod n plus m tau by mod n plus mod m that ratio has to be bounded below by something positive it cannot go to 0 all right.

So, it is kind of geometrically very acceptable if you look at this picture, but but one needs to actually show that this is true because tau is tau is an in the upper half plane. So, it is it is a non real quantity. So, this is the small exercise which I will ask leave to you to do and lets believe this exercise then and then.

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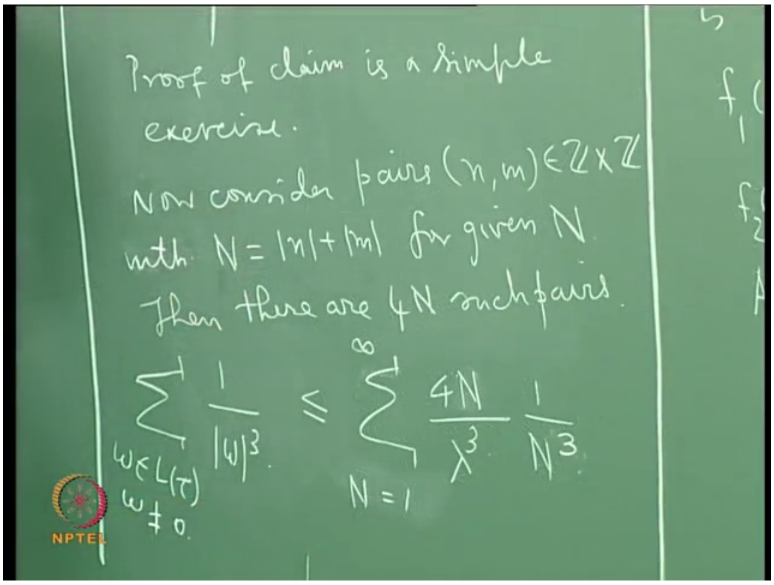

So, proof of claim is a simple exercise which I will ask you to do and, but let us apply it. So, you know if now consider now consider pairs n comma m which are integers and n is mod n plus mod m for given n. So, look at these quantities actually this the this there is a double summation here I mean I mean there are 2 there are 2 freely I mean there are 2 integers here, but I want to group it into one.

So, what I do is I look at all those pairs n comma m such that mod n plus mod m is accepting capital n then how many do I get I will get 4 capital n such pairs because you know for every pair n comma m the n comma minus m is also another pair and then minus n comma m is also another pair and then minus n comma minus m is another pair. So, you actually get 4 of them and therefore, there are total 4 n such pairs. So, I should say now consider pairs with this, then there are or well 4 n such pairs there are 4 n such pairs and therefore, you see there are 4 n such pairs. So, you see if you if you look at if you try to sum over. So, if you try to sum over one by mod omega cube if you try to you sum over one by mod omega cube were where omega is in there is varying in the lattice omega not equal to 0 then you see this is this is less than or equal to all right 4 4 by. So, so let me let write that down 6 sigma omega in L of tau.

So, now I will change it to a summation n equal to 1 to infinity, all right, I will get 4 n 4 by 4 by lambda cube 4 n by lambda cube no I will get 4 n 4 by lambda cube into one by n cube into one by n square right because. In fact, I will get 4 n by lambda cube into one by n cube this is what I will get and you see and then of course, this n cancels and. So, essentially what I get is I get I can I can pull the 4 by lambda cube out and I will get sigma n equal to one to infinity one by n squared which you know converges and mind you see, this n above cancels with an n below. So, it should have been n cube here if it had been n squared I will get summation one by n and that is going to diverge and why did I get n cube because if you go back I let this I let this omega I let this Z on top remain as it ok.

If I had replaced this Z by mod omega naught, then I would have lost this power of 3 here and I would I got a power 2 and then when I came back here I would have I would have just shown that this is dominated by sigma 1 by n that is not going to be of any use. So, it is very purpose full that this mod Z was left as it is all right. So, what does this calculation tell you this calculation tells you that the. So, this calculation tells you that the series given by S of Z this calculation tells you that this series given by S of Z converges uniformly and absolutely on all compact sets and of course, at of course, you should not look at it at you are not worry, I mean you are not looking at the points which are poles namely the lattice points outside the lattice points this series converges.

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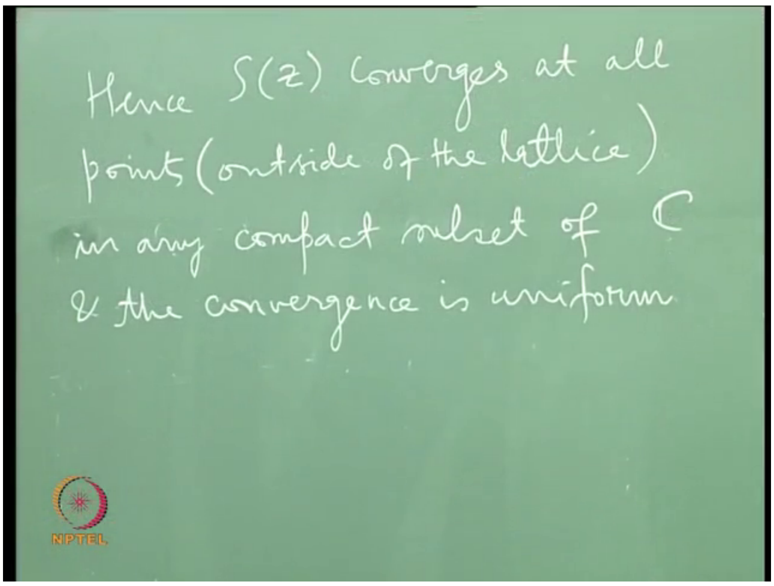

So, hence S of Z converges at all points outside of the lattice in any compact subset of C and the convergence is uniform.

Now, because of the uniformness the resulting function is also going to be analytic. So, you do get a function which is analytic accepted points of the lattice. So, then you have to worry about the at the points of the lattice you have to say that each point is a pole of order 2 you have to say that and then you have to say that there is this resulting function here you have to say that it is W periodic then you also have to then it will be more or less clear that because of this theorem because of this theorem it will be more or less clear it will be it will be clear that this is this has to be a Weierstrasss phi function.

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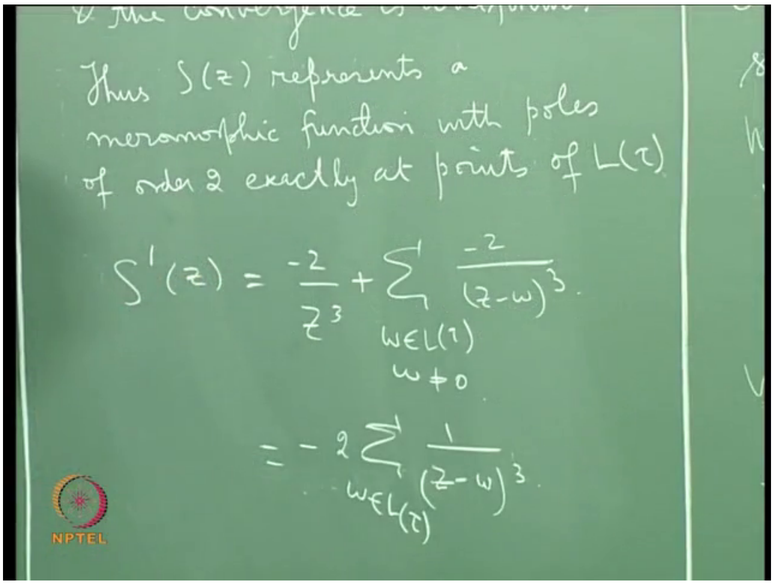

So, let us do that thus S of Z represents meromorphic function with poles of order 2 exactly at points of L of tau because you see if you look at points of L of tau see the same calculation will tell you that you know as Z goes close to omega you can see that. So, long a Z is not omega this series of course, converges, but because of this term the limit Z tends to omega will become infinite.

So, it will certainly be a pole and if you if you take limit EZ tends to omega of EZ minus omega the whole squared into S of Z you will see that that will be a non 0 quantity therefore, Z equal to omega each point of the lattice will certainly be your pole of order two. So, so it is certainly a meromorphic function with pole of order 2 at each of those points now the next thing, I want to tell you is that this is in fact, W periodic. So, the trick is as follows see you calculate the derivative of S of Z because of uniform convergence you can do term wise differentiation. So, you do term wise differentiation and what you get you will get minus 2 by Z cube plus summation over omega in the lattice omega not equal to 0 of, I will if I differentiate this I will get minus 2 by Z minus omega the whole cube and I can write this compactly as minus 2 sigma 1 by Z minus omega the whole cube well with omega in the lattice.

I can also include I can also includes Z equal to 0 there I in the original explanation I had to remove Z equal to 0 term because I had this one by omega squared, but here that is gone. So, what I get is S prime of Z is this and this term is because this is the summation over the whole lattice if you if you translate is it by a lattice element this sum is not going to change therefore, this is certainly W periodic this is certainly W periodic function. So, let me notice that.

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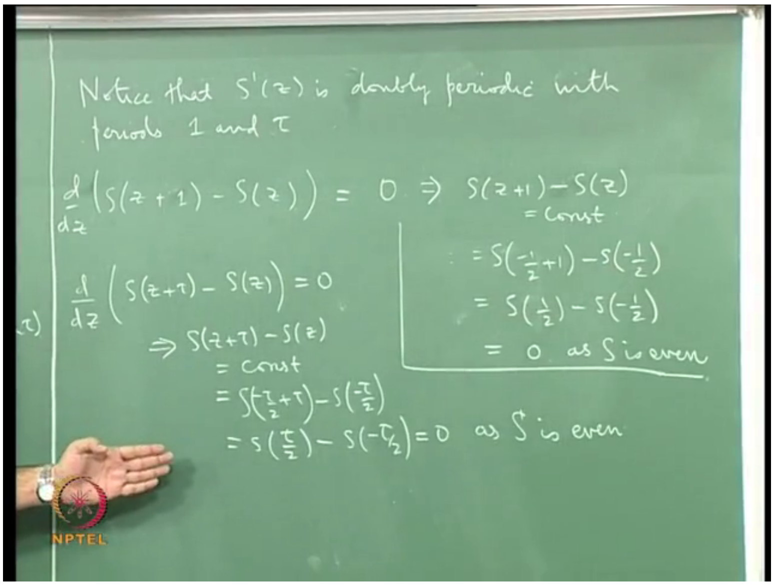

S prime of Z is periodic is W periodic with period 1 and tau, it is certainly W periodic with periods one and tau that is because you take an element of the lattice and if you translate the lattice by an element of the lattice they are going to get back the lattice. So, nothing is going to happen.

So, what is the upshot of this; if I take S of Z plus. So, suppose I take S of Z plus 1 minus S of Z suppose, I consider this difference S of Z plus 1 minus S of Z then and suppose I differentiate it, I will end up getting d by d Z of this will be 0 because I will get S dash of Z plus 1 minus S dash of Z, but S dash is P has period 1 and tau. So, this will be 0. So, what this will tell you is that you know S of Z plus 1 is equal to S of Z I mean minus S of

Z is a constant and that constant has to be 0. Now that constant has to be equal to S of minus half plus 1 minus S of minus half which is S of half minus S of minus half and this has to be 0 because if you look at the expression for S from the expression for S its very clear that S is an even function wherever it is defined.

So, what we have proved is that S of Z has 1 as a period the same kind of argument will show you that S has also tau as a period because d by dZ of S of Z plus tau minus S of Z will be 0 and that will imply that S of Z plus tau minus S of Z is a constant and now I can plug in minus tau by 2 and. So, that constant has to be S of minus tau by 2 plus tau minus S of minus tau by 2 and this is S of tau by 2 minus S of minus tau by 2 and that is again 0 as S is even. So, let me write it here also as S is even S is even because of its definition mind you we have not yet proved that S is W periodic. So, you have to be careful here ok.

So, what we have proved is therefore, S is W periodic and it is a meromorphic function with poles exactly at exactly at the points of the lattice pole of order 2 and if you look at the expression for S its clear that the singular part at the origin is one by Z squared.

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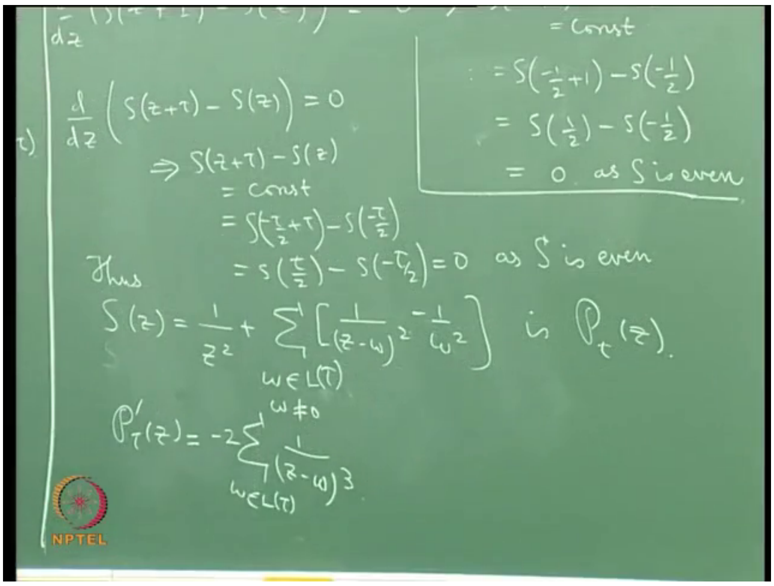

So, so let me write that down S of Z dash S of Z is 1 by Z squared plus sigma omega in the lattice omega not equal to 0 one by Z minus omega the whole squared minus 1 by omega squared is . So, let me say it in words it is the; it is W periodic with periods one and tau it has a pole of order 2 at each of the lattice points and if you look at the origin you can see that this is going to be analytic in a neighborhood of the origin and the singular part is one by Z squared and it is also clear that the constant term in the Laurent development at the origin is 0, therefore, it satisfies all the conditions of the theorem that I stated and therefore, this has to be the Weierstrasss phi function. So, is is P tau ok.

So, the upshot of all these is that you also have a nice formula for the derivative of the Weierstrasss phi function which is going to be very important for us in the forthcoming lectures because that is going to tell you why the this why any complex torus is actually algebraic it is going to help us to explain that. So, let me also write that down the derivative is just minus 2 sigma omega belonging to L of tau 1 by Z minus omega whole cube. So, I will stop here.