An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1 -dimensional Tori and Elliptic Curves

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Lecture - 29
The Necessity of Elliptic Functions for the Classification of Complex Tori

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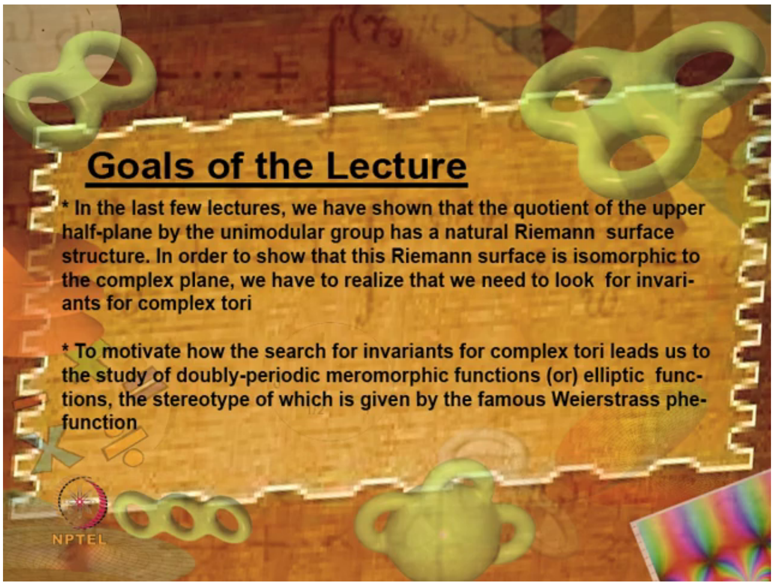

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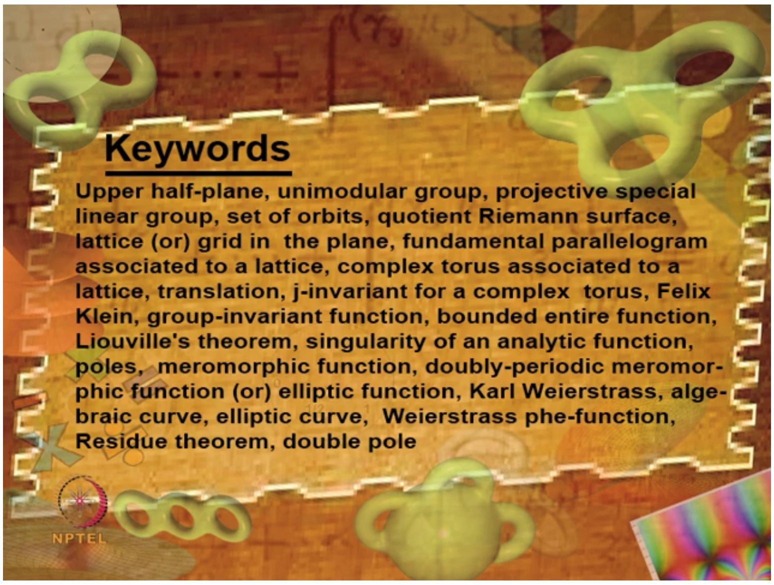

So, let us recall whatever general aim has been so far, see we are trying to look at the set U mod PSL 2 z and in the last few lattice we have proved that U mod PSL 2 z is a Riemann surface, and how did we prove that we proved that by show by realizing that PSL 2 z is you know a discrete group, and it leaves invariant the upper half plane. So, it is a fiction group and then we proved the result that for a fiction group, you know the notion of discreteness coincides with the notion of being claniun and because it is claniun. Therefore, we prove that the set of points where the group acts properly discontinuously namely the region of discontinuity of the group is certainly contains the invariant half plain or disk, ok.

So, in this way PSL 2 z, just using the p the fact that PSL 2 z is discrete and the fact that PSL 2 z is leaving the upper half plane invariant, we are able to conclude that the you know the region of discontinuity for PSL 2 z includes the upper half of plane, and therefore, you are able to divide by the upper half plane and conclude that U mod PSL 2 z is a certainly a Riemann surface. Now the next part of the story is to try to show that this Riemann surface U mod PSL 2 z is actually none other than the complex numbers. So, I just want to show that there is a holomorphic isomorphism of U mod PSL 2 z with the complex numbers. So, to try to understand where this will lead to, let us try to do some heuristic thinking.

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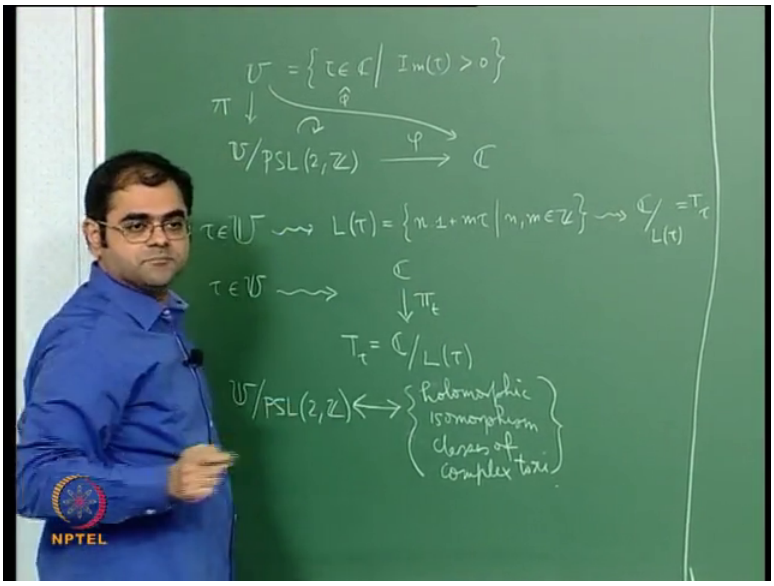

So, you see basically you have. So, you have U you have U mod PSL 2 z and so you. So, I may be I put a thickened U to specify that this is the upper half plane alright and of course, U the set of all complex numbers tau, such that imaginary path of a tau is positive which is the upper half plane and you know PSL 2 z leaves upper half plane it is precisely the set of holomorphic automorphisms of the not precisely. In fact, it is a subgroup of PSL 2 r, which is precisely the automorphism holomorphic automorphisms of the upper half plane. And we want to prove that this Riemann surface we have proved that this is this map is a map of Riemann surface it is it is a holomorphic map, and the quotient is a Riemann surface. We want to show that this is Riemann surface is actually biholomorphic that is isomorphic holomorphic isomorphic through the complex numbers ok.

So, for a moment believe that we have that, what does it mean? See for that matter suppose I give you even any function phi into the complex plane; suppose I give you a function phi from this Riemann surface U mod PSL 2 z this is the set of orbits of PSL to z in the upper half plane, which has been made into a Riemann surface. Suppose I give you a holomorphic function on this well you must realizes that if I compose it with this projection, suppose I call this projection as a pi. If I compose it with this projection, I get a holomorphic function let me call this as ph hat, I get a holomorphic function from the upper half plane to z, because after all this is just composition if this is a holomorphic map canonical quotient map is holomorphic, because that is the way we constructed the Riemann surface structure on this and I have assumed that phi is a holomorphic function. So, the composition if I call it as phi hat, then phi hat is also a holomorphic function.

So, but what kind of a holomorphic function is it? Is a holomorphic function that is going to have this same value on the orbits of PSL to z in U. So, trying to find even if holomorphic function for that matter even trying to find a set theoretic function on this Riemann surface, is trying to find a function on the upper half plane which is constant on orbits, it has to have the same value on orbits, only then I a function on the upper half plane which is constant on orbits only such a function will go down to give define a function on the set of orbits. So, you see. So, you are trying to look at you say in particular if you want a holomorphic function into z, that is a holomorphic map into z you are trying to look at holomorphic maps from U to z which are constant have the same that is which have the same value on every PSL 2 z to z orbit. So, what does this mean?

See, let us go back and try to understand what this set U mod PSL 2 z was originally where it came from. See if you remember what we did was, for every tau in the upper half plane, we associated the lattice L of tau, this was you can take the lattice spanned by one and tau. Namely you take all complex numbers of the form n times 1 plus m times tau, where n and m are integers this is the lattice this is that is it is an additive subgroup of the complex numbers. And of course, it is a z module, it is a discrete sub module z sub module of the complex numbers. This is a lattice and then if you remember using this lattice, we constructed the torus the complex torus defined by the lattice by simple going z modulo this lattice.

So, from this we went to C modulo this lattice to get a torus T sub tau. So, this T sub tau it was a complex torus. So, you if you remember C to you know C mod L tau. So, you know here when Ic mod L tau L of tau is elements in this lattice, but I am thinking of them as acting on C by translations. So, and of course, translations are certainly mobius transformation. So, L of tau is identified with a subgroup of automorphisms holomorphic automorphisms of C and then I am just simply going modulo of that subgroup and you know what this is. This is you know that this is a covering; it is a universal covering for this complex torus. And of course, the fundamental group of this of the base of this complex torus can be identified with the depth transformation group, which is precisely L of tau which is a isomorphic to L of tau which is isomorphic to z cross z ok

So, we are we know this. So, the point is for every tau in U we are associating this this complex torus and in fact, what we how did we get this U mod PSL 2 z, we noted that to the complex tori complex one dimensional tori T tau 1 and T tau 2 are holomorphically isomorphic if and only if tau 1 and tau 2 can be moved to each moved by an element of PSL to z. In fact, what we proved was that the set U mod PSL 2 z namely the orbits of PSL to z in U is in bijective correspondence, with the set of holomorphism holomorphic isomorphism classes of complex tori. So, U mod PSL 2 z is a let me put this is a bijective correspondence with a set with the set of holomorphic isomorphism classes of complex tori ok.

So, this is how we started and therefore, you see if I go back to this situation, trying to get hold of a function on U mod PSL 2 z is the same as trying to get hold of a function that does not change if you change the equivalence class of the element here under PSL 2 z. In other words it should be a something that does that depends only on the isomorphism class, the holomorphic isomorphism class of the complex torus that is defined by a point above. In other words what is a function like this? A function like this which does not change on orbits is called an invariant. So, of course, you know it is invariant under PSL 2 z, but geometrically you can also think of it as trying to produce for every isomorphism class; holomorphic isomorphism class of complex torus, you are trying to produce something which depends only on the holomorphic isomorphism class ok.

So, you must think of. So, you must understand that trying to cook up a function like this has to has got to do with finding invariants for holomorphic tori. Namely quantities mathematical quantities which depend only on the holomorphic isomorphism class of the torus. You are trying to find for every complex torus some quantity that quantity should be such that if you change the torus by an isomorphic torus, the corresponding quantity should not change such quantities are called invariants and you are trying to find invariants for tori. So, it is obvious that you know these invariants have got to do with the geometry of this torus. And in fact, they have to they must have something to do with the geometry of all the tori which are in that particular orbit under PSL to z, namely all the tori which are holomorphically isomorphic.

So, this leads to trying to you know look at some trying to study these tori themselves. So, well you know if you go back to the famous ideas of Felix Klein. So, he somehow thought that geometry has to be a geometry of a space has to be dictated by the functions on the space somehow if you know the all the functions on your space, then you should know the geometry on the space and you can turn this around and say that you can dictate a geometry on the space by specifying the kind of functions that you are allowing on the space. And in a way this is we can see this already in trying to say that you know if you want to study differentiable manifolds, you take the allowed functions to be z infinity functions. If you want to study complex manifolds you take you take the allowed functions to be holomorphic function well, if you want to study algebraic manifolds you take the allowed functions to be locally rational functions for questions of polynomials.

So, if you go by that then I would like to look at functions on this, and see this is certainly a Riemann surface. I would like to look at let us say holomorphic functions of this.

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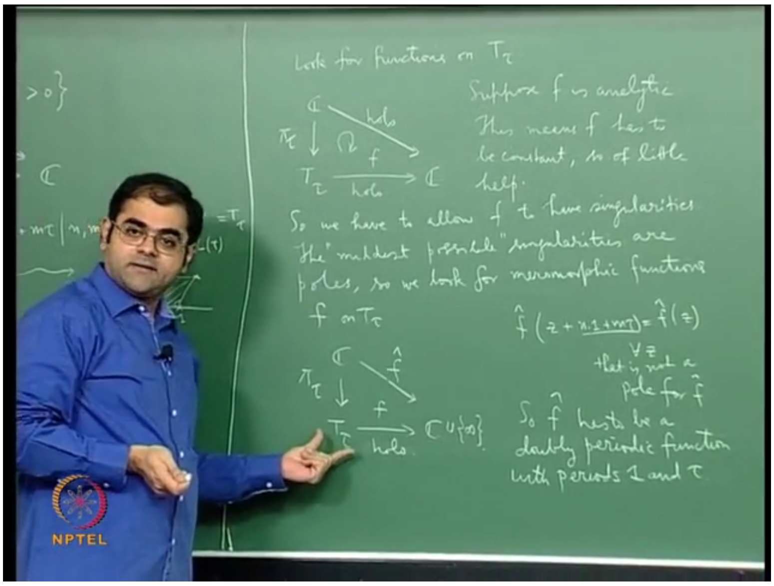

So, what I would do is well I will try to look for functions on the torus, you try to look for functions on the torus and then you know one hopes that you try to find the function that depends only on the isomorphism class, the holomorphic isomorphism class of the torus and then you have found an invariant. And finding such invariants will precisely try to give you functions from this set into whatever set.

For example, trying to show that there is an isomorphism phi of this set with c. Let us say I want to show even that there is a bijection of this with C suppose I just want to show leave alone holomorphic isomorphism. Suppose I just want to show that there is a bijection of this with C, then what I should do is for every whole isomorphism class holomorphic isomorphism class of a torus, I have to produce for you one complex number and actually you can produce one such number that is called the j invariant of the torus and we will see that ok.

So, but the point I am trying to say is that trying to look for invariants like this, leads you to try naturally to look at functions on the torus now well you see. So, you see suppose I look at function on the torus. So, suppose I look at a function let me call this as f let us let us assume that you are looking at a complex suppose you are looking at a complex function and let us be optimistic and let us assume that well they the best thing to assume is that f is analytic. So, suppose f is analytic. So, I am searching for functions on the torus and suppose I have function like this which is analytic well let me call this projection as pi sub tau not to be confused with this pi. So, well I can compose like this. So, I put a circle at (Refer Time: 16:42) I put a circular arrow saying that this arrow is this followed by this, that is what it means here also this arrow is this followed by this that is what this circular arrow means namely that the diagram commits.

So, well you can see that if this is analytic or holomorphic then this is holomorphic, but this is not going to give us anything that is because you see this guy here is compact this this this torus topologically is compact. Because you know basically it is the image of the fundamental parallelogram formed by that is the fundamental parallelogram, the grid formed by 1 and tau. So, actually what is happening here is that if you remember see I have one here and I have tau there ok.

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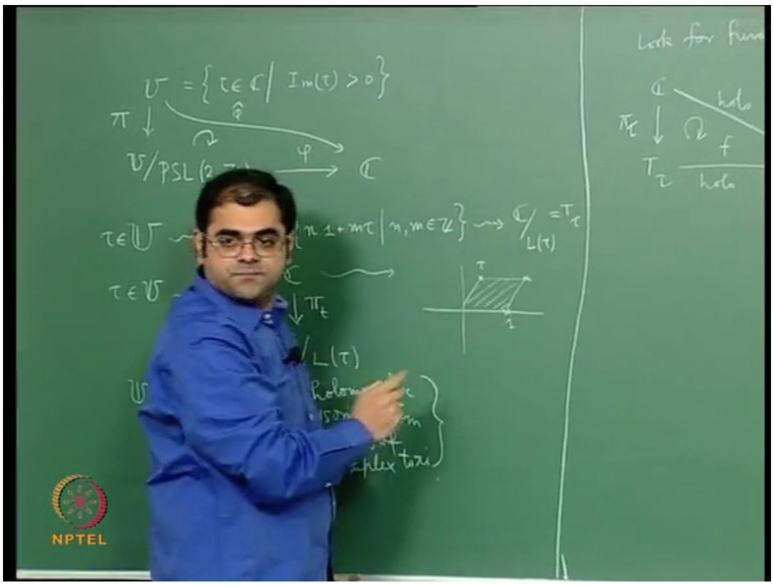

And you know there is a parallelogram like this, and if I repeat this parallelogram to form a grid the points of the grid are precisely the point of this lattice. So, this the points of the grid of the size is the lattice and that the image of this this whole parallelogram along with the boundary (Refer Time: 17:59) onto the torus, and this is of course, compact it is closed and bounded and therefore, the torus is also compact, because this map is continuous after all you have given it the (Refer Time: 18:06) quotient topology.

So, this is compact therefore, and of course, a holomorphic map is continuous therefore, the image of this set in z is going to be a compact subset of z in particularly it is going to be bounded; therefore, this map this holomorphic map will be an entire function which is bounded and then it has to be constant. So, we are not going to get anything. So, the moral f the story is trying to look for analytic functions on the torus, is not going to help. So, what does it mean? It means you have to allow singularities. So, let me write that here this means f has to be constant, of little help. So, we have to allow f to be to have singularities, you must allow functions to have singularities.

So, well you know if you what are the singularities possible singularities on an analytic function, well you know there are isolated singularities and there are non-isolated singularities. The isolated and among the isolate singularities there are the removable singularities which are actually not singularities, then there are the poles and then there are the essential singularities.

So, if you want the mildest type of singularities what you do is that, you would like to have at least poles and trying to study analytic functions, which have only poles and if you also ensure that these poles are isolated leads to the study of what are called as meromorphic functions. So, the whole the simplest kind of functions you can try to think of on the torus are going to be meromorphic functions. So, the mildest possible singularities are the poles, and so we look for meromorphic functions f on the torus, you look for meromorphic functions. What are meromorphic functions? Meromorphic functions are functions, which are analytic at all points except for a subset of isolated points, where this function has only poles and well.

So, you are looking at meromorphic functions alright and notice that; and therefore, what is it that I can do is well I can have a discrete subset and I can have poles at each point with certain multiplicities, but even in that the simplest thing I could have is pole only at one point. So, of them you take a pole at only one point for example, alright. So, well, we could look at meromorphic functions with pole at only one point or let us say pole at a few points to begin with. Now what I want you to understand is, what does looking at such a function means above in the complex plane. So, you see supposed I have an f which is meromorphic and well. So, if it is a meromorphic functions it is not defined at the poles.

So, in principle when I draw this arrow this is not really a map from this to this, this is a map only from this into z union infinity, because at poles the function will take the value infinity. So, you know I will just. So, the best way to put it is well. Let me put z union infinity and then write it as n holomorphic. So, you know meromorphic functions into meromorphic functions complex valued functions are precisely the holomorphic functions into z union infinity, because what happens is at the at the poles you define the function value to be infinity, and then you get holomorphic functions into p 1 the Riemann sphere. So, this is z union infinity the (Refer Time: 23:17) complex plane, which is identified with the Riemann sphere by the stereographic projection, it is a Riemann surface and then a hole up into that is the same as giving a meromorphic function otherwise; and the poles will be all the inverse image of the point at infinity ok.

So, mind you I am looking at meromorphic functions on the torus, now what is it above what is it that is happening above. So, you see the first thing is that suppose I assume that there are the pole; suppose I assume that the poles are isolated, then if I go all the way above and look at this function f hat, then this function of f hat will also be a meromorphic function, the only thing is that the they the poles will just it will have infinitely many poles, and what will happen is if you have if f has a pole here, the inverse image of that pole will be an inverse it will be a point, which will be spread over a grid and then that grid will be translates this grid. And it will have poles at all those points and what is. So, what you get is looking at good looking at a meromorphic functions on the torus, is looking at a meromorphic functions on z and further notice that this function of f hat has to be invariant under the action of this of translations by elements of these lattice.

So, you see f hat of z n times 1 plus m times tau must be equal to f hat of z for all z that is not a pole for f hat. This should happen because after all these 2 points are going to be identified when you by this map by this projection, by this because this point is a translate of that by this element, that is an element of the lattice and that is the group of translations by which you are modding out. So, this function f has to satisfy this condition. Now you know see what does it tell you what it tells you is that it tells you that f hat has to be a w periodic function with periods 1 and tau. So, f hat has to be a w periodic function with periods 1 and tau ok.

So, what is the moral of the story? The moral of the story is the simplest kind of functions you can try to write on the torus are going to be w periodic meromorphic functions, and these are precisely what are called as elliptic functions, and where studied first by Weierstrass who gave a decent beautiful theory for these functions. And it is through these functions that you are completely understand the geometry of the tori and everything that is connected with tori right from number theory to algebraic geometry is done by looking at the meromorphic functions. So, the after this point our discussion has told us, that you know actually you should look at w periodic meromorphic functions on C namely elliptic functions on C they are called elliptic functions and with periods 1 and tau.

So, well let us function let us look at the properties of such a function see.

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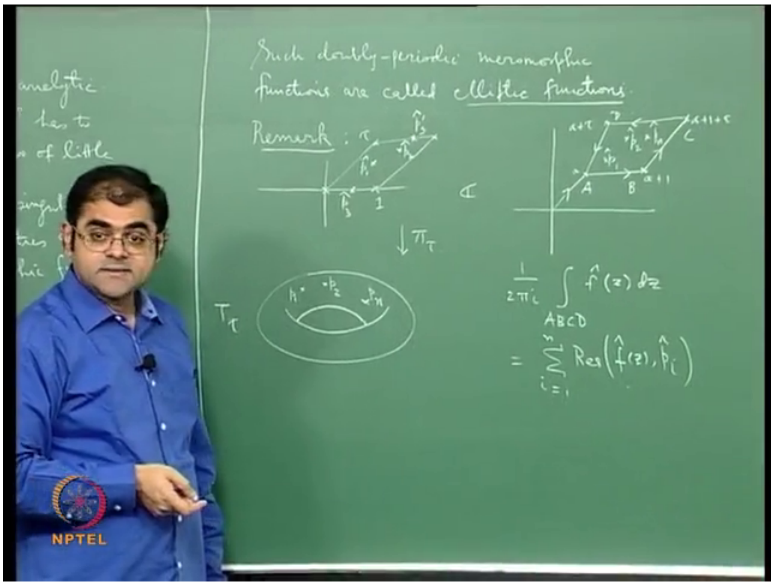

So, such functions such functions such w periodic meromorphic, and why they are called elliptic functions is because, they help to actually prove that this Riemann surface this complex torus can be thought of as zeros of an algebraic equation and that. And in fact, it turns out to be an algebraic curve, and that curve is called an elliptic curve and that is the reason why they are called elliptic functions.

So, I will come to that later on because it is a deep fact that every complex one dimensional torus of this form is actually an algebraic is given by an zeros of algebraic equations and in fact, it is a much more deeper theorem, that you take any Riemann surface, you put the topological condition that the Riemann surface is compact, then amazingly the Riemann surface this compact Riemann surface becomes can be thought of as zeros of an algebraic functions. So, it becomes completely algebraic. So, it is a demonstration the demonstration of that fact for the case of complex tori can be achieved by using the elliptical functions ok.

So, we have (Refer Time: 29:37) studied particular case of elliptical functions, which are called famous Weierstrass phi function. So, I will explain you how this Weierstrass phi functions is motivated. So, well we have to look at elliptic functions alright. Now what are the kind of functions that we can look at. So, of course, you have to decide let us take at least one point or a couple of points, and try to and let us decide that the function has poles at those points, but before that I want to make a remark.

So, you see here is a remark if I calculate. So, you know I take. So, here is my. So, let me draw a small diagram. So, you see I have. So, here is my torus. So, here is my torus and you know and what is above is a complex plane, and this is the projection, this is T tau see. Suppose I take a say a couple of points finite set of points separated from each other, and take their inverse image, well you know there is grid of on the complex plane there is this lattice, which corresponds to 1 and tau.

So, I have this lattice here, this is the fundamental parallelogram whose image is here. And if I take the image of such points the inverse image, well if the inverse image is if you look at inverse images if an inverse image is going to lie inside this parallelogram in the interior, then it is going to be only one. But it could happen that you know it could lie on the boundary in which case you will get 2 and if it is one of the vertices you will get all the 4. So, what I am trying to say is that since there are only definitely many points. So, let me call these points as say p 1 p 2 etcetera up to, let us say p n. Then what I can do is well I can translate this grid in such a way, that you know I can translate this grid in such a way that all these inverse images of these points, I have only one representative inside that grid I can translate it ok.

So, I can move it if some points tries to come at the vertex I can avoid it. So, I can translate this. So, in other words what I can do is you know well, it may happen that well p 1 is inside, p 2 in inside maybe for p 1 I can find a unique representative for p 2 maybe I can find a unique representative because it is inside, but maybe for p 3 I may I might get 2 representatives one here and one on this, one exactly in the in the opposite edge of the parallelogram well. So, it will be some p 3 prime hat and then maybe pn might be might have all the 4 vertices as universal images it could happen alright. But well the point is I can move this move this this parallelogram a little bit by translating it, so that all the representatives there are come into the interior of the parallelogram and all the representatives are already there.

So, you know I can actually choose an a such that you know if I take a I can choose a vector I can choose a complex number and trying to think of them a vectors well I can choose a complex number a, such that you know if I now draw this parallelogram if I translate it by this a, then when I will get a plus 1 then I will get well a plus 1 plus tau, then here I will get a plus tau I will get this parallelogram, which is just a translate of that, but I can choose a in such a way that I get exactly one representative for each point in this line inside in the interior I can do this ok.

So, choose an a such that this translated parallelogram has exactly one translated (Refer Time: 34:26) from each of these things. And look at let us look at let us call this this boundary. Let us give an orientation to the boundary of this parallelogram so that it becomes a closed path and then try to look at the integral 1 by 2 pi i integral over where I mean let if I call this rectangle as I mean this parallelogram as A B C D if I take the path integral over ABCD of this function, which is above which is f hat of z dz if I look at this, what will I get? Well you know by the residue theorem, I should get the sum of the residues at each of the points inside where the function has poles, ok.

So, what I will get is I will I get summation i equal to one to n residue of f hat of z at the point p i hat. This is what I will get this is the residue theorem, but you see the point is that f hat is periodic is w periodic. Therefore, the integral of f hat from A to B will cancel with integral of f hat from C to D. Because after all the f values here are the same as f values here, because tau is a period and the integral of f from B to C will cancel with the interval of f from D to A that is because one is also a period for the function. See in the in the definition of the periodic function I could have taken m equal to 0; what it tells is that f of z plus 1 is also f of z and I could have put n equal to 0 and I will get f of z plus tau is also f of z. Points here it differ from the from the corresponding points there by tau and points here differ from corresponding points here by 1therefore, all the integrals will cancel this is going to be 0.

So, the moral of the story is the sum of the residues at this poles is going to be 0 what does it tell you? That tells you that to begin with you cannot have a single pole which is a simple pole, because suppose there is only one pole and the simplest thing is that I would like to have only one point where there is a pole and the simplest kind of pole I can think of is a simple pole, I can try that that is the simplest I can try, but that will not work; because then will tell you that residue at that pole is at a simple pole is 0 that cannot be. So, what is the moral of the story the moral of the story is that you must have if you take only one point, you must have a pole of at least order 2 or you should at least use 2 points with simple poles. So, there are 2 choices you can make to at the simplest trying to study the simplest meromorphic functions. Either you can choose a single you can choose a single pole and make it order 2 or you can choose 2 poles and of order 1 in any case you must have some of residues and both the poles must be 0 ok.

So, Weierstrass did is that he chose a single pole of order 2 and. So, what we do is well we look at this probably this line does not look. So, very straight let me draw another. So, that it looks little decent ok.

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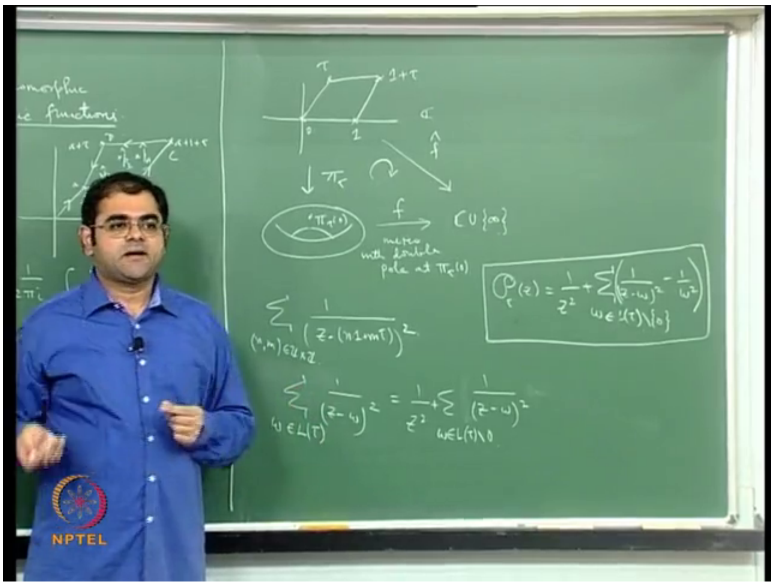

So, what we will do is we will do the following thing; we will take this lattice I mean. So, we go back to our.

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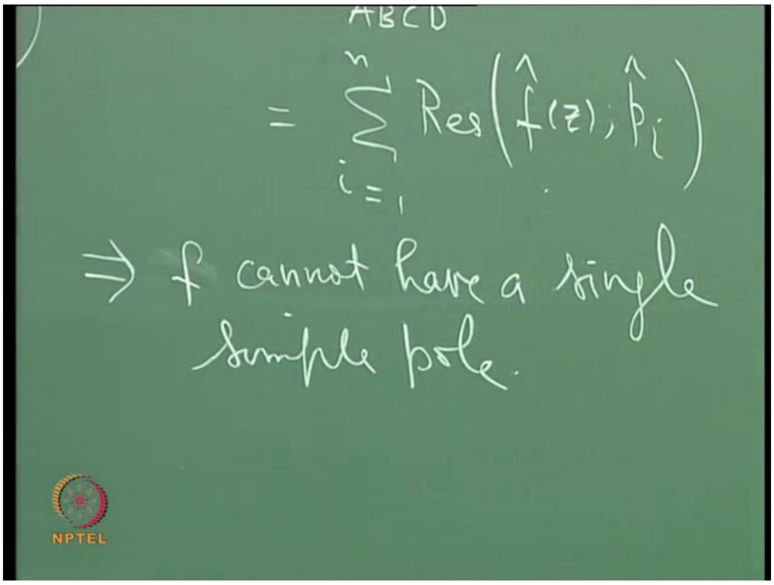

So, let me write the conclusion here this implies we cannot f cannot have a single simple pole, this cannot happen alright. So, here we take this. So, here is my again here is my lattice. So, this 1 this is tau, this is 1 plus tau just a complex thing and. So, well here is my torus T tau and I am looked.

So, I am we are following Weierstrass, prime to get hold of trying to study meromorphic functions, having a double pole at one point below. So, and what is that one point we may actually choose that one point with the image of this whole grid. So, this whole grid goes to some point here, because all of them are equivalent under translations of the lattice. So, all the points of the grid are going to be mapped to a single point, that point you can look at that point and well I can. So, in particular it is that point is a image of 0. So, I will call that point as pi tau of 0, it is also pi tau of any point of the lattice, and I am trying to study a function defined on the torus, which is meromorphic with a double pole, which means a pole of order 2 at that point pi tau of 0, this is what I am going to study.

And well; that means, that the function above f hat is going to be an elliptic function, a meromorphic w periodic function, which is going to have a double pole at every point of this lattice. It is going to have a pole of order 2, at every point because you know if this function has a pole here, then they this function above will have a pole at every point which is the inverse of this point. So, your f hat will be a w periodic function which will have you expected to have a double pole at every point of the lattice. What kind of a what kind of function would it look like? Try to guess see you can write well you can just take 1 by I can take if you know if I take a point of the lattice, it looks or the in the form n one n times 1 plus m times tau and if I write 1 by z minus n times 1 plus m times tau this will give me a pole of order 1 at n times one plus m times tau ok.

But if I want to want a pole of order 2, I cannot have a simple pole at one point, I need to have a double pole from what I said here. So, if you want to have a pole or order 2 I square alright and then this. Well, this gives me pole of order 2 at this lattice point, but then I want this to happen at every lattice point. So, what I do is I sum this over n comma m integers, but of course, I should not let n comma m equal to 0 comma 0, well I would rather well n comma m can be 0 comma 0 also I mean that is not a big deal, I will get one z squared, I will get a pole at the origin that is all fine, but then. So, you know this summation, because the summation let if I write it as omega in the lattice, 1 by z minus omega the whole square this is what the summation becomes ok.

Now, there is a problem, I will tell you why. See this is 1 by I can write it as 1 by z squared I separate the pole at the origin alright. So, I will write it a sigma omega belonging to the lattice minus the origin, 1 by z minus omega the whole squared. So, all I have done is I have just removed at the pole at the origin. Now we see of course, I have written down some series here and of course, there are questions of convergence right now as z tends to omega alright of course, if I fix a particular omega and I let z to tend to omega then of course, this is going to go to infinity, that it should because it is a pole, but then you see I get the term 1 by omega squared, and you do not want for reasons of convergence what you do is that you take away that 1 by omega squared.

So, what you do is put minus. So, let me write that separately. So, this is exactly due to Weierstrass. So, you have the Weierstrass phi function, phi tau of z is 1 by z squared plus sigma omega not a non zero element of the lattice defined by tau, belonging to L tau minus 0 of 1 by z by omega the whole squared minus 1 by omega squared. Now this minus 1 by omega squared is plugged in. So, that as z tends to omega the contribution of 1 by omega squared, that comes from this term gets cancelled with the contribution minus 1 by omega squared that comes from the other terms, and it is there for convergence purposes.

So, this is called; in fact that is there is a bracket here. So, this summation is common to both of these guys and this is the Weierstrass phi function. So, this is how the Weierstrass theory of elliptic function starts, this is how it begins and it goes on to give beautiful results about to tori, which are connected with complex geometry algebraic geometry number theory and so on. So, this is the motivation for studying the Weierstrass phi function.

Now of course, one needs to check that whatever one has written down heuristically is actually something that make sense you cannot simply write the series and expected to be a function, you have to check that it converges, you have to check that it is really a meromorphic function, you have to check that is a meromorphic function that converges. Namely; it is analytic at the points where you expect it to an analytic namely it is analytic at every point different from any point of the lattice. And you make sure that it is w periodic. And you must also make sure that the only poles of this function or at the lattice points and the poles are of order 2. I mean nothing is clear; I mean everything has to be proved. But the point is that one is able to arrive at the phi function in this way heuristically, and then it is a matter of some analysis to prove that one is indeed right.

So, I think in the forth coming lectures, therefore what I will do is I will show that this phi function indeed has all the properties. And we will see that this phi function actually characterizes the torus thoughts. So, by that I mean if you take two non holomorphically isomorphic tori the phi functions you get they will not be compatible with each, you will get different phi functions. So, this phi function kind of distinguishes. And this is a starting point of trying to find an invariant for tori and which is what we are trying to search for. We are trying to find a complex invariant for every isomorphism holomorphic isomerism class of tori. So, this is the starting point.

So, we will continue with the forth coming lectures.