An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-dimensional Tori and Elliptic Curves

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Lecture - 03
Maximal Atlases and Holomorphic Maps of Riemann Surfaces

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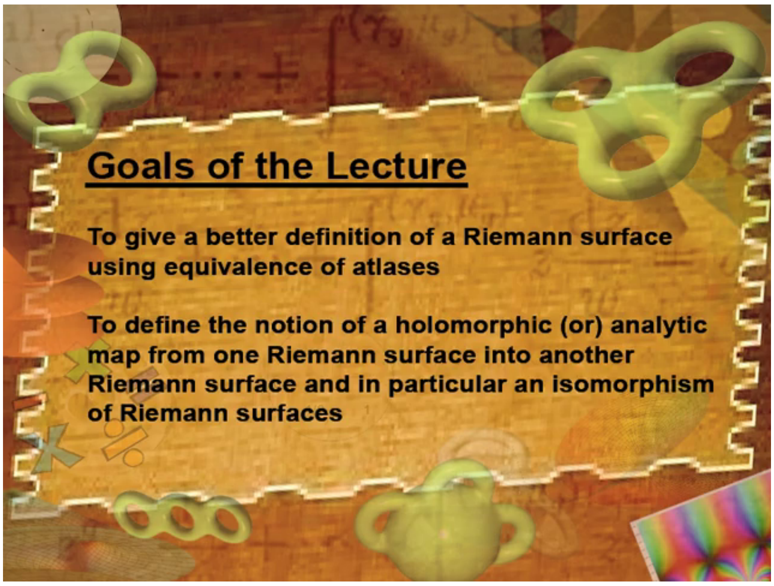

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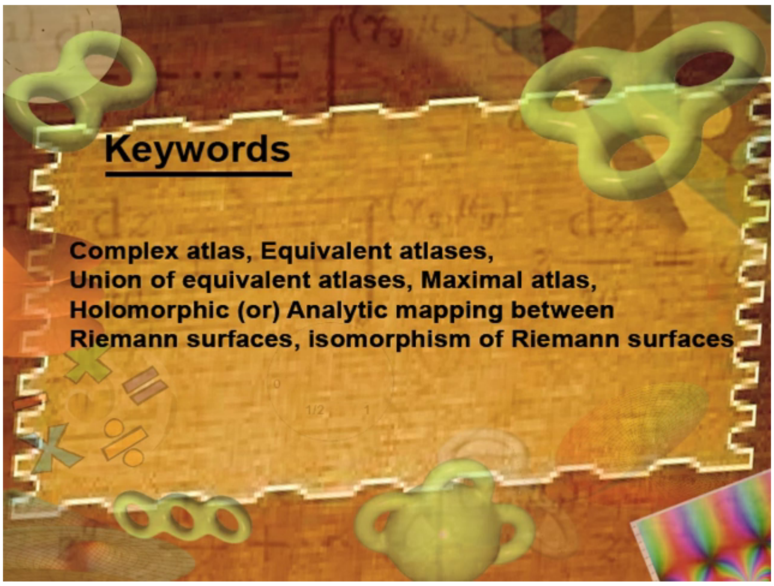

So, let us get along with this third lecture in the series on Riemann surfaces and algebraic curves. So, let us begin by recalling what we wanted a Riemann surface to be. The aim is first of all I want to make our definition of Riemann surface slightly more sophisticated. So, you recall that we took a real surface, so we start; we start with the real surface X.

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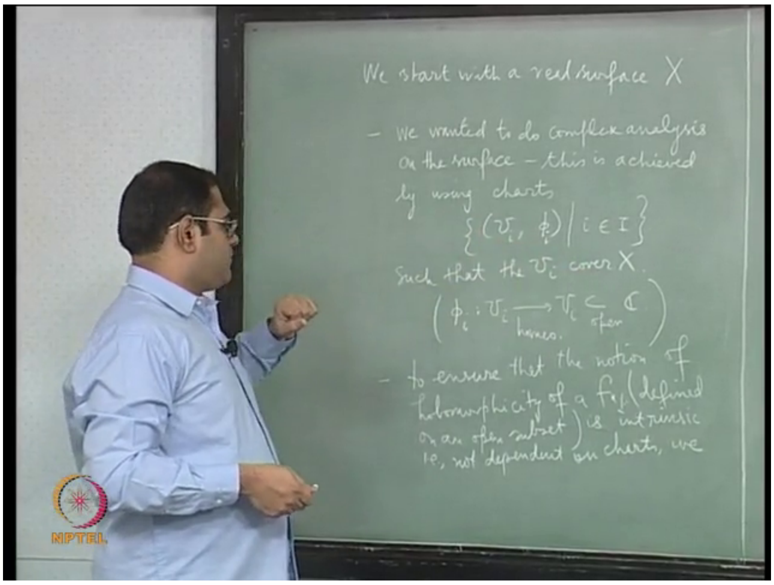

So, this is a surface which for the moment is one that we can visualize in three space for example, the plane or the cylinder or a torus or the sphere these are all surfaces that you can imagine in R 3 in three space.

And what is it that we wanted to do we wanted to do complex analysis on the surface. So, we wanted to do complex analysis on the surface on the surface and that essentially was trying to do the following if you are given an open set on the surface, and if you are given a function defined on that taking complex values I would like to decide clearly when I can call this function as holomorphic. So, because complex analysis is all about studying the properties of holomorphic functions or analytic functions, well in order to do this I told you that we have to use what are known as charts. So, so this is achieved this is achieved by using charts U comma phi U i comma phi i if you want where i belongs to an indexing set such that the U i cower X.

So, basically if you recall phi i was a homeomorphism from U i into V i which is an open subset of the complex plane and this was a homeomorphism. So, this was our definition of chart. And then for example, if I wanted to decide whether if a function defined on U i is holomorphic all I had to do was to go from V i to U i by taking the inverse map which is defined because V i is homeomorphism and follow it by my map my function. So, I get a function from an open subset of the complex plane into the complex plane for which I can certainly define what holomorphic is which I already know.

So, we could use these charts to define when a function is holomorphic and we of course, needed to cover every point on the on the surface. So, we needed cover of the surface by charts. And well there was one problem that we needed to avoid and that is that when we decide the holomorphics key of a function it should not be something that depended on the choice of a chart because holomorphicity of a function should be an intrinsic property of a function and therefore, it should not happen that the function is holomorphic with respect to one chart and it is not holomorphic with respect to another chart because charts can intersect.

So, we overcome this problem by requiring that the charts are compatible. So, the second condition was to ensure that the notion of holomorphicity of a function defined on an open subset is intrinsic, is intrinsic to the function that is not dependent on the chart on charts we required that the charts were pairwise compatible.

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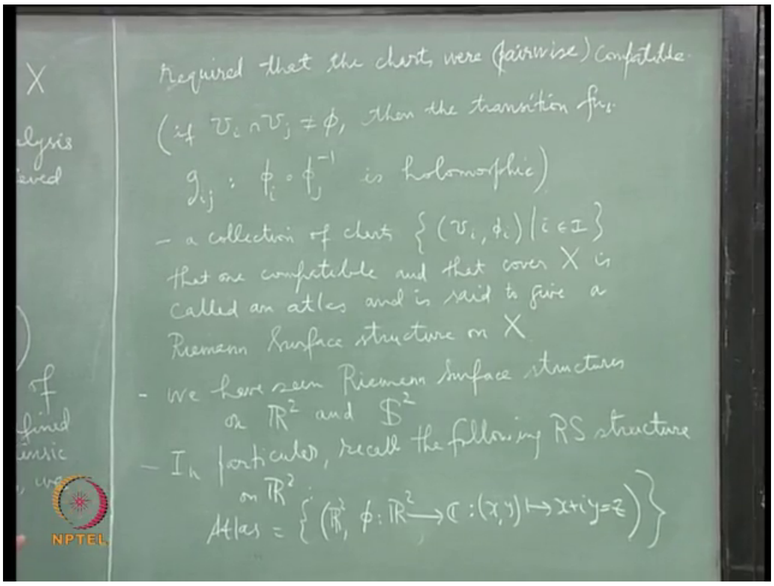

And what was this pairwise compatibility? The pairwise compatibility condition was if that if U i and U j were intersecting non trivially that is the intersection was not empty. Then the transition function the so called transition function, so called transmission transition function which I denote by g ij which is first apply phi j inverse and follow it by phi i.

This transition function is going to be defined it is going to be a homeomorphism from an open subset of the complex plane to another open subset of the complex plane it is a homeomorphism and I want this to be holomorphic. And that is, and because its injective that is also equivalent to requiring that this function is a the holomorphic isomorphism because an injective holomorphic map is also an holomorphic isomorphism may they inverse also becomes holomorphic, so is holomorphi.

And well. So, once we are given a collection of charts which cover X and when all these charts are pairwise compatible then we call this an atlas and we say that X along with that atlas is now a Riemann surface right. So, a collection of charts U i comma phi i, i belong to I that are compatible which means pairwise compatible and that cover X is called an atlas and is set to give a Riemann surface structure on the real surface X.

So, this was our definition of what Riemann surface should be you take a real surface cover it by charts which are compatible and this collection of compatible charts is called an atlas and real surface along with this atlas put together is called as a Riemann surface, you can rather call it a Riemann surface structure on X.

Well I just want to make this definition a little bit more sophisticated in this lecture in the following way. So, well after I gave this definition we have seen in the last lecture how we can give Riemann surface structures on the plane and on the sphere. So, we have seen we have seen Riemann surface structures on R 2 and S 2 this is a real plane and this is the real sphere we have we have seen examples of that.

Well and I want to in particular look recall your attention to the to the natural Riemann surface structure on R 2 that makes it the complex plane, the usual complex plane. So, you see in particular. So, let me write that in particular recall the following Riemann surface structure. So, I will just abbreviate Riemann surface to R S, so that I avoid writing it out in all the time and I can save some time.

In particular you recall the following Riemann surface structure on R 2. So, the atlas is just consisting of a single coordinate chart. And what is the chart? It is just the open set is the whole of R 2 and the map. So, let me make way for some more space the map phi from R 2 to C is just the natural identification namely it takes X comma Y to X plus iy which is z. So, this is a natural map and I am just taking a single chart.

Now, this chart of course, covers the whole plane and it is an atlas the collection consisting of only this chart is an atlas because there is no compatibility condition that has to be verified. So, by logic if that is a condition that does not need to be verified it is deemed to be true vacuously true. So, this indeed an atlas and what it does is that it makes R 2 into the complex plane c. So, we say that the complex plane C is a natural Riemann surface structure on R 2 and where the notion of holomorphic function is the usual notion of holomorphic function that we study.

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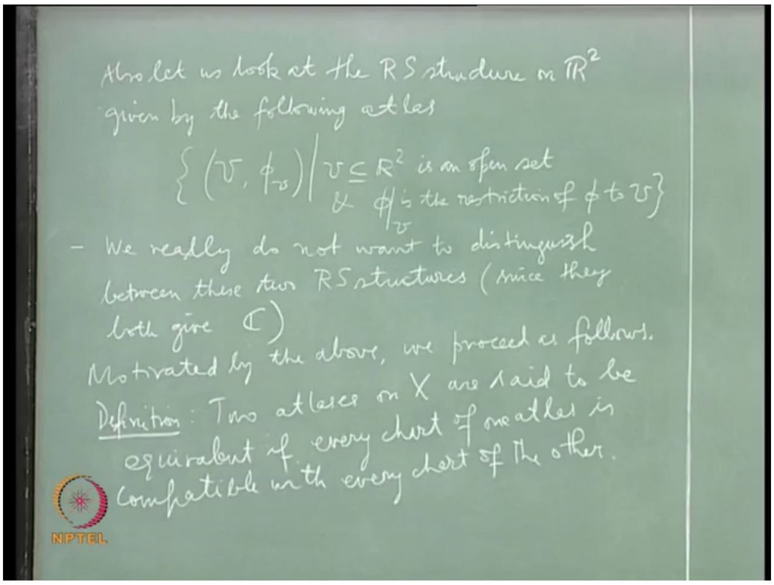

Now, also let us look at the Riemann surface structure on R 2 given by the following address. So, here is my atlas my atlas consists of all possible U comma phi restricted to U where U in R 2 is an open set and phi is a phi restricted to U is the restriction of phi to U, where phi is defined as I have done it here namely the natural identification.

So, you look at you look at this collection. Of course, you can see that this is a this contains this because I can take U equal to R 2 and then I have that chart as well, but then now I have so many charts I can write as many of them as there are open sets in R 2. And well in principle it is very clear that this is also going to give you just the complex plane you are not going to get any other Riemann surface structure on R 2 its again the same complex plane.

So, we really do not want to distinguish between this the Riemann surface structure given by this complex atlas and the Riemann surface structure given by this complex atlas we really do not want to do that. And for that we just make the definition a little bit more sophisticated. So, this is the motivation for making the definition more sophisticated. So, what we do we really do not want to distinguish between these two Riemann surface structures since they both give they both give C the both of them give the complex plane. And why is it that we say that they give the same Riemann surface structure because you take a function on R 2 define an open subset of R 2 its holomorphic with respect to this structure if and only if its holomorphic with respect to that structure because holomorphic with respect to any of these structures is just holomorphic in the usual sense. So, really there is no difference in deciding whether a function is holomorphic.

So, you see this goes in tune with a philosophy of Felix Klein the great German geometry who said that the geometry of a space is controlled by the functions you allow on that space. So, if I look at the holomorphic functions given on the Riemann surface given by this by this structure they are no different than the holomorphic functions given by the Riemann surface structure on this by this atlas. So, essentially they should be the same space that is the motivation.

So, what do we do to what we put into the definition to make sure that we do not really distinguish between such things? So, we do the following thing what we do is that, you take two possible atlases which give Riemann surface structures on a given surface and then you define them to be equivalent if every chart in one atlas is compatible with every chart in the other you put this equivalent this condition. So, motivated by the above we proceed as follows. So, definition two atlases on X are said to be equivalent if every chart of one atlas is compatible with every chart of the other.

So, we define this equivalence or this is basically a definition of equivalence of atlases complex atlases it is a very simple definition it says every chart of one atlases compatible with every chart of the other atlas. And you can see that this is clearly an equivalence relation because this is symmetric about the two charts. And well there is something else that you can also see you take two such equivalent atlases and take their union then you will find that that again gives you an equivalent atlas which is bigger than both of them because an atlas is just a collection of charts which are mutually compatible.

So, if you have two atlases which are equivalent; that means, every chart in this atlas is equivalent every chart in the other atlas if you put them together you will still get an address because the condition for an atlas is just compatibility. So, it is very clear that if you have two atlases you can put them together and you get new atlas. And then now the Riemann surface structure given by any of these atlases and their union should all be the same you should not really distinguish between these.

So, the moral of the story is that you should try to change the definition in such a way that you include in an atlas as many compatible charts as you can. So, we have this notion of what is called a maximal atlas right. So, if there are two compatible atlases I can put them together take their union I get a bigger atlas and then in this way I can keep on enlarging the atlas until it becomes maximum.

Now, a standard argument using Zorn's lemma in algebra, rather set theory will tell you that a maximum atlas will always exist. So, this is a Zorn's lemma argument and that will tell you that given any atlas I can find a maximal atlas which contains this atlas and it will also tell you that the maximality will also tell you that this maximal atlas is unique. And then I am in good shape because I can now define the definition I can now define a Riemann surface to be one that is a real surface that is equipped with the maximal atlas. And once I say that then you know there is no difference between the Riemann surface defined by this and the Riemann surface defined by this because both of them will have the same maximal atlas. So, that is what I am going to write down now.

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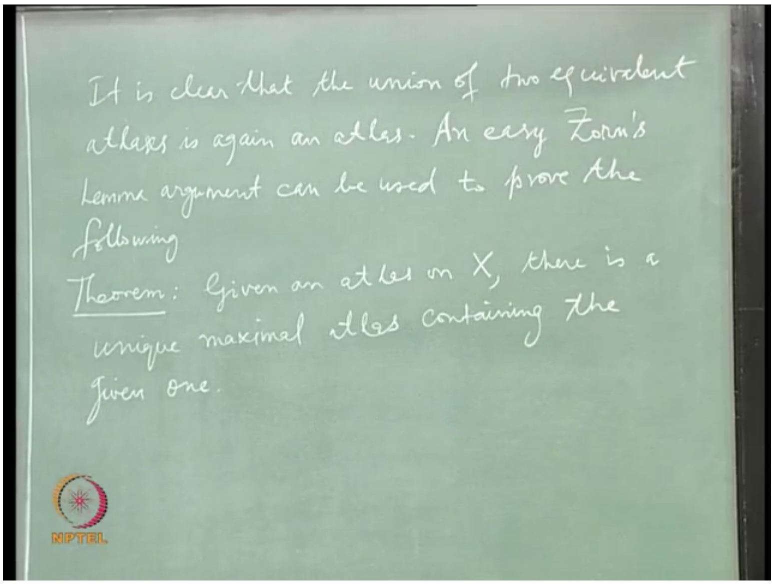

It is clear that the union of two equivalent atlases is again an atlas an easy Zorn's lemma argument can be used to prove the following, prove the following. So, let me call this as well theorem given an atlas on X that is a unique maximal atlas containing the given one.

So, the Zorn lemma argument is usually applied in the following sense you have a partially ordered set and then you verify the condition that every chain in that partially ordered set has an upper bound and then Zorn lemma will guarantee that maximal elements will exist.

So, in this case the set is the set of all possible atlases on X and the partial order, order is just containment and atlas is said to be lesser than another atlas if every element of this atlas is also an element of the other atlas, it is by inclusion. And if you give me a chain of atlases then it is obvious that the biggest one is an upper bound or even if the chain is infinite I could simply take the union and that will be an upper bound. So, every chain has an upper bound and now Zorn lemma wills will assure you that maximal elements exist. So, you can find maximal atlases.

So, because of this theorem I can now define a Riemann surface structure to be one specified by a maximal atlas, but that is just its of technical significance because it allows you to identify these two Riemann surfaces being the same that is the advantage. But in practice it is not a big deal because when we specify a Riemann surface structure we are just going to give an atlas and then we are going to assume that the Riemann surface structure is the one that corresponds to the maximal atlas which contains our given atlases. So, for all practical purposes we will work with some atlas which is convenient for us it need not be the maximal atlas. But this maximal atlas is just a condition that I put into the definition of Riemann surface so that I really do not have to distinguish between the Riemann structure surface structure here and here. So, let me write that down.

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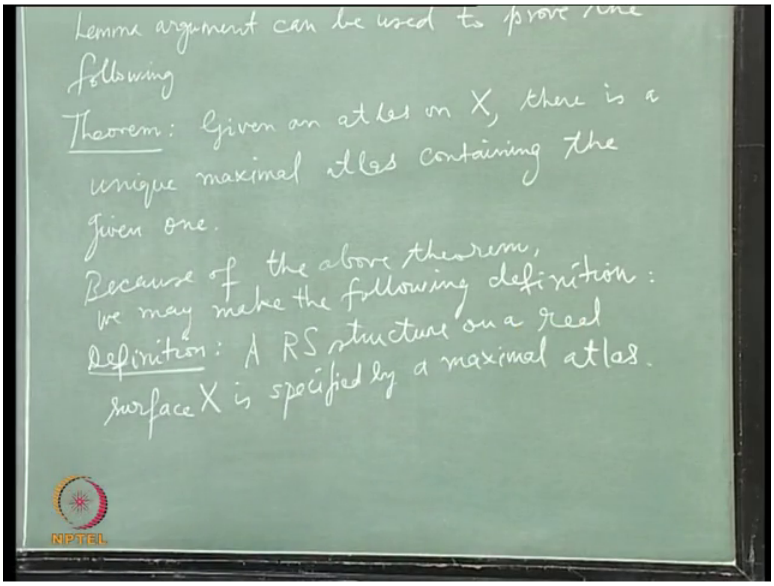

So, let me just complete this. So, here is a definition a Riemann surface structure on a real surface X is specified by a maximal atlas. So, here is my revised definition. So, if you want to define a Riemann surface structure on a real surface you take the one given by a maximal atlas for all practical purposes you would only take any atlas that is suitable for our use and then we will say that we are referring to the Riemann surface structure given by the maximal atlas which contains the one that we have specified.

So, let me repeat the advantage of this definition is that I do not have to distinguish between this Riemann surface and this Riemann surface just because they are two different atlases, the surfaces are the same. I mean both of them give the complex plane and I do not want to there is no point in distinguishing between them.

So, let us get back to these examples that I gave the last class. So, you see our say, I was trying to give examples of Riemann surface structures and what I had in mind are of course the real plane and well the sphere the real sphere, then I also have in mind cylinder and then the torus. So, these are all objects that we these, these are all surfaces that we can easily think of in R 3.

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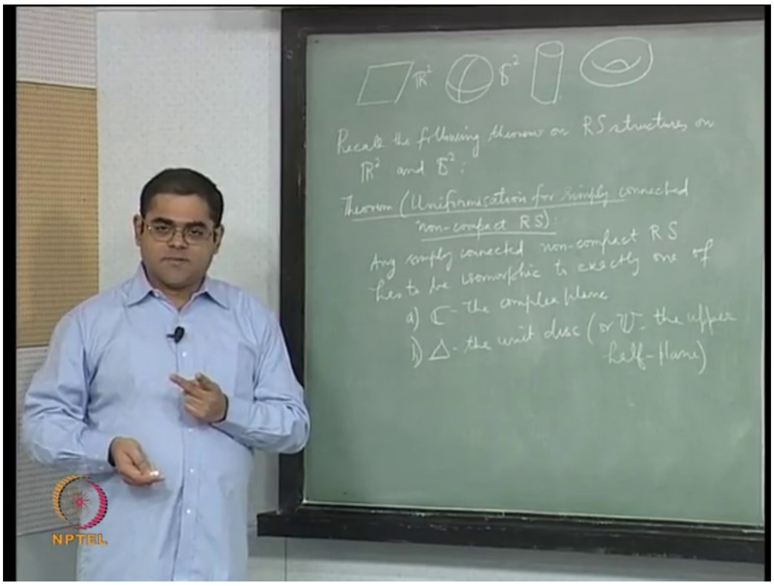

Well, we have already looked at the case of the plane and the sphere. So, let me recall that case because there is something there that I have to formalize a little more. So, recall the following theorems on Riemann surface structures on R 2 and S 2. So, that was this theorem which is called as the uniformisation theorem for simply connected non-compact. So, this is the uniformisation theorem for simply connected non-compact Riemann surface, which says the following any simply connected non-compact Riemann surface has to be isomorphic to exactly one of a C the complex plane, b delta the unit disk or U the upper half plane.

So, this is the information theorem for simply connected non-compact Riemann surfaces. Take a Riemann surface which is simply connected, so simply connected means that any closed loop on the surface which is a continuous image of the interval can be shrunk continuously to a point. So, that is that just says that there cannot be any holes in the surface. And you take a simply connected Riemann surface I assume that is also non-compact and then the uniformization theorem says that it has to be either isomorphic to C or it has to be isomorphic to delta, delta is unit disk and you know you can always find a by holomorphic map a mobius transformation in fact, which can map delta to the upper half plane you can map any disk into any half plane you know that, so yeah. Instead of delta I could have also said U, U is up upper half plane namely complex numbers with imaginary part greater than 0.

Well, this uniformization theorem for simply connected non-compact Riemann surfaces. And this is this tells you what does it tell you, it tells you that if you try to look at Riemann surface structures on R 2 because R 2 is certainly non-compact and it is certainly simply connected. Then on R 2 you can put only two possible Riemann surface structures one isomorphic to C which is given by the natural identification, then the other one is isomorphic to delta and that was example 2 of the previous lecture. And the Riemann mapping theorem ensures that these two are not isomorphic these two are not equivalent they are not by holomorphic.

So, let me also recall the corresponding uniformization theorem for that applies to the real sphere. So, well let me first rub this off.

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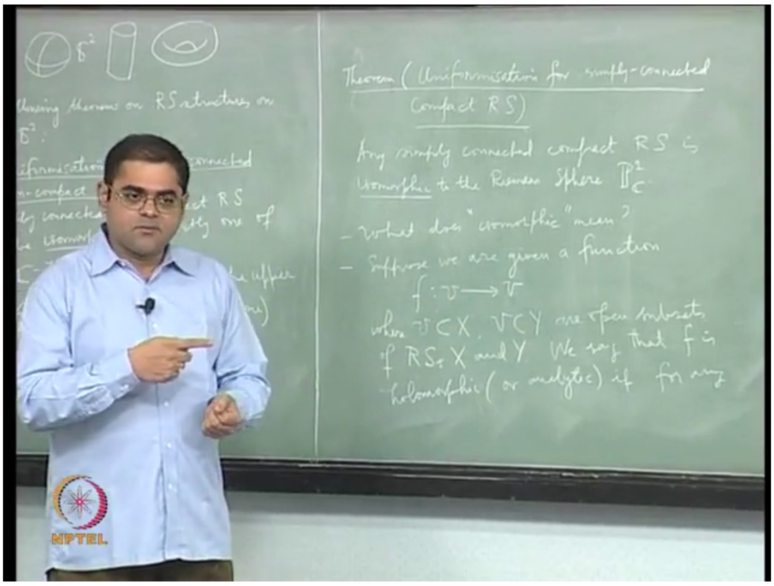

So, theorem uniformization for simply connected compact Riemann surface, so what does this say? It says any simply connected compact Riemann surface is isomorphic to the Riemann sphere which I will denote as p 1 of C, I denoted as p 1 of C for reasons I will explain later. And this example of the Riemann, the this example of a Riemann sphere was the example of Riemann surface structure on a S 2 which it was the last example in the previous lecture and basically the 2 charts were given by taking the 2 open sets to be the sphere minus the north pole, and this sphere minus a south pole and the coordinate maps were given by the stereographic projection on to the plane.

So, and then you can also mention I do not I hope you have checked it that the transition function is just given by z going to 1 by z and that is of course, holomorphic when set is not equal to 0. Of course, you will have to compose one of the stereographic projections with the complex conjugation to get the transition function correct. And from this I deduced or rather you can easily read deduced that if I try to put different Riemann surface structures on S 2 I am not going to succeed I am going to get only 1. All the Riemann surface structures I try to impose an S 2 I am only going to get 1, no matter what collector what choice of charts or atlases I use.

Well these are all these theorems are all are not easy theorems the proofs are little involved, but eventually we will we will try to prove them in the course. Well the reason why I recalled these two theorems is to draw our attention to the following thing which I have to formulas because I said you see. For example, here that any simply connected non-compact Riemann surface has to be isomorphic to exactly one of the following here also I say any simply connected compact Riemann surface is isomorphic to the Riemann surfaces to the Riemann sphere

So, I am here talking about an isomorphism between Riemann surfaces that is something that I have not really defined, but it is very intuitive and you will see that it is very easy to define. So, the idea now is I am going to try to define when a map from an open subset of Riemann surface to another open subset of another Riemann surface is holomorphic.

So, let us go to that. So, let me write this what does isomorphic mean. So, we proceed to formalize this. So, suppose we had given a function f from U to V, where U is U is an open subset of X and V is an open subset of Y and X and Y are Riemann surfaces. Suppose we are given a function from an open subset of one Riemann surface 2 an open subset of another Riemann surface when we call this holomorphic. So, you see that is the first thing that I like to define. Once I define this then I can define when a map from X to Y itself is holomorphic and then I can define when a map from X to Y is an isomorphism by requiring that it has to be holomorphic and the inverse map also is holomorphic. And that again of course, it will follow that if its holomorphic and if it is bijective then the inverse map will be also holomorphic.

So, that is the reason why in order to formalize the notion of isomorphism I have to formalize this notion of a holomorphic map between open subsets of Riemann surfaces. And again well how do we do this it is again by using the charts all right. So, so let me make the definition as follows. So, maybe I will draw a diagram it should not be it should be easier for you to visualize it with a diagram. So, let me draw one.

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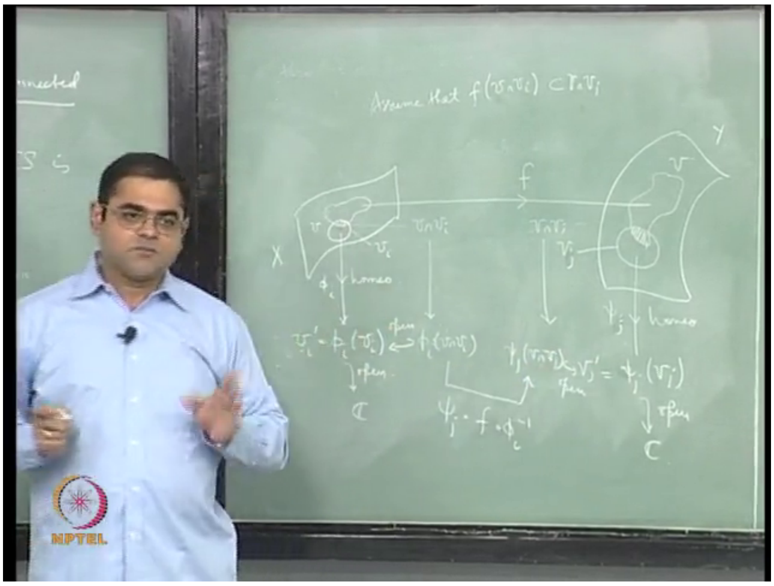

So, here I am, here is one Riemann surface here is my Riemann surface X, well and here is another Riemann surface say Y, well and I have I have this open set here U in X and its some other open set here which is say V in Y and well I have this function f and the aim is I want to say that this function is holomorphic. So, what do I do, well I take any chart here on this surface which intersects U. So, what I do is that well I take a set here which is the domain of a chart. So, let me call this chart as U sub i and well there is going to be a homeomorphism phi i from U i into an open subset of the complex plane. So, let me write that also let me call this as a phi i of U i let me call this as I do not want to use V. So, let me call this as well U i prime if I want and this is an open subset open subset of C and if U i comma phi i is a chart on X that intersects U; that means, U i intersects U and this is the intersection.

Similarly, I take a chart on V which intersect a chart on Y which intersects V. So, that is again a pair. So, that is another chart which I will call as V i this is the domain of the chart and then the chart also consists of a homeomorphism I call this as psi i, I will not I not the same I well let me use something else j if you want coming from a different index index set. In fact, I could do away with the if is and js, but anyway since I written it let me keep them. Well this is again a homeomorphism of V j on to V j prime which is the image of V j under psi j which is again an open subset of C and this is again a chart which intersects V domain as chart the chart is basically consisting of two data though. The first one is the domain of chart the second is the homeomorphism that that makes the domain look like an open subset of the complex plane.

So, well what I can do is that you see I can do the following thing, this shaded region here is just U intersection U i and that U intersection U I will go to an open subset, subset phi, phi i of U intersection U i that is again going to be an open subset of this because under homeomorphism the image of an open subset is again an open subset and restricted to that open subset it is still a homeomorphism all right. And similarly if I call this, this intersection of V j with V as I mean it is V intersection V j and well under the homeomorphism psi j its going to this open set psi j of V intersection V j that is again an open set open subset of V j V prime j.

Now, what I can do is I can go from this open set to this open set by using the map f namely I apply phi i inverse and I land in this intersection then I apply f restricted to that intersection right and of course, I do assume that the image of that intersection under f does meet some part of this intersection, so that I can compose. So, I can look at this map from here to here and this map is just going to be apply phi i inverse then apply f and then apply psi j assume that f restricted to U intersection U i or rather f of U intersection U i goes into V intersection V j.

I can assume this here of course; I am assuming the ref is continuous. Well now when I do this I now get a function from an open subset of the complex plane to another open subset of the complex plane and well I can easily decide if this function is holomorphic. So, the point is that even if this condition is not satisfied then do not verify anything verify this only when this condition is satisfied and all possible charts here U i comma phi i and all possible charts that V, V j comma psi j do it and if it is going and if this composition is going to be holomorphic then I declare that f is holomorphic.

So, it is really it looks a little complicated when I say it the first time, but actually I can say it in a nutshell by saying that well to decide whether a function is holomorphic all I do is that I write it in terms of local coordinates and check whether it is holomorphic because the moment I like I write it in terms of local coordinates it means that I am using the coordinate charts to get a mapping from open subset of C 2 and another open subset of C and there it is easy to decide when a function is holomorphic.

So, let me write that down ah. So, let me write the following we say that f is holomorphic of course, the other word that is always used is analytic or analytic if for any charts, chart U i comma phi i of X any charts and well V j comma psi j of Y such that U i intersection U is non empty, V j intersection V is non empty, the composition the composition, psi j circle f circle phi i inverse is holomorphic whenever f of U intersection U i goes into C intersection V j.

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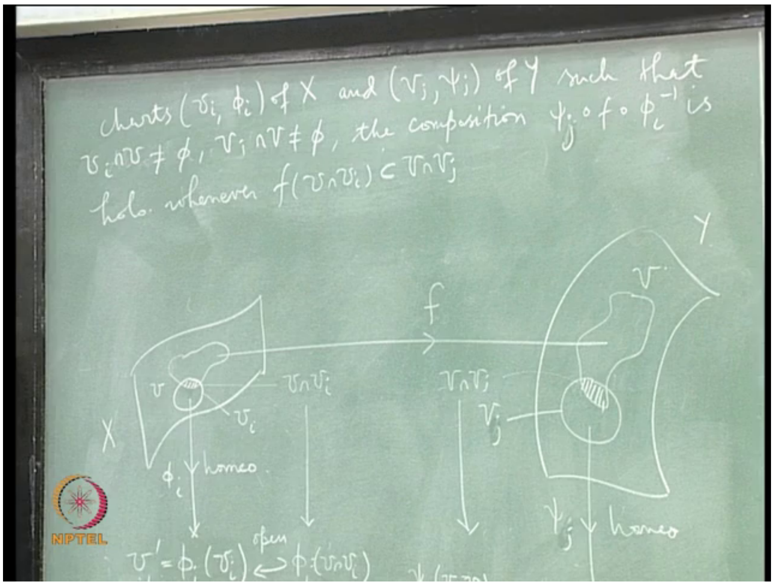

Here of course, I am assuming the ref is continuous to decide if a map between an open sub subset of one Riemann surface and an open subset of another Riemann surface is holomorphic I just have to decide using the local coordinates and then you can see that of course, this notion of holomorphic map being holomorphic is intrinsic.

So, it is not really going to its not going to be an ambiguous definition and well we are going to say that to Riemann surfaces are isomorphic if you are able to find a map which is holomorphic and which has an inverse that is holomorphic and that is in this case also going to be enough to require that its holomorphic and it is bijected. So, well, so that is when we call the Riemann surfaces are isomorphic and that is the isomorphism I am referring to in these two statements. So, that is that is what I wanted to clarify.

Well now you see having done this there is one immediate, there is one immediate observation that we can make which will help us in the following sense.

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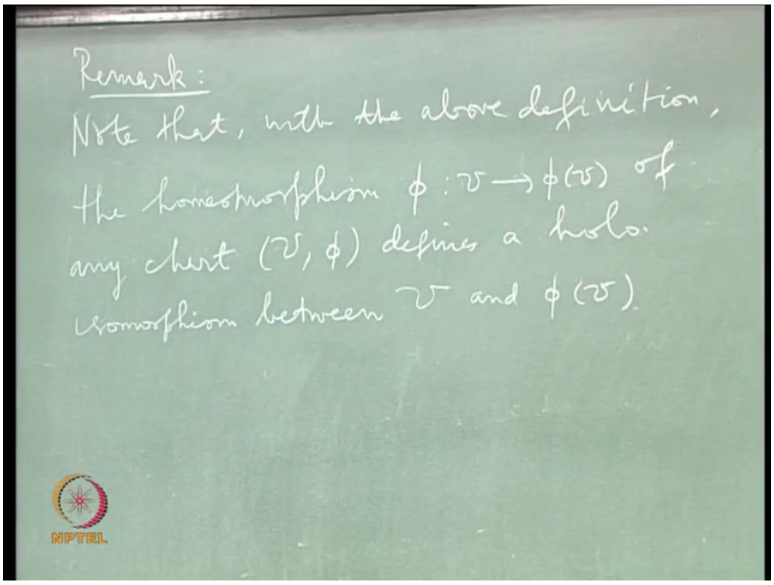

Note that with the above definition. So, this is this remark actually it pertains to a point raised by one of the students in I think it this was in the one of the previous lectures that you know that, if you take any chart that is going to give you a function from an open subset of X to an open subset of C, now you see X is a Riemann surface and C is also a Riemann surface and basically we required this chart on this function only to be homeomorphic, but with this definition it actually becomes holomorphic.

So, it is rather clever the definition of holomorphicity automatically makes the function of every chart a holomorphic isomorphism. The homeomorphism of any chart defines the holomorphic isomorphism between U and phi. You think of U, U is an open subset of X, phi U is an open subset of C and V are in this situation we have a map from an open subset of X into an open subset of Y. So, Y is now C.

So, the point is that all your coordinate charts all the mappings in your coordinate charts they are all holomorphic. And now this gives you another view of what a Riemann surfaces it is got by gluing together open subsets of the complex plane at the point I want to make is that a Riemann surface basically is obtained by gluing open subsets of the complex plane and the gluing is done by the transition functions So, that is the remark that is what this remark tells us.

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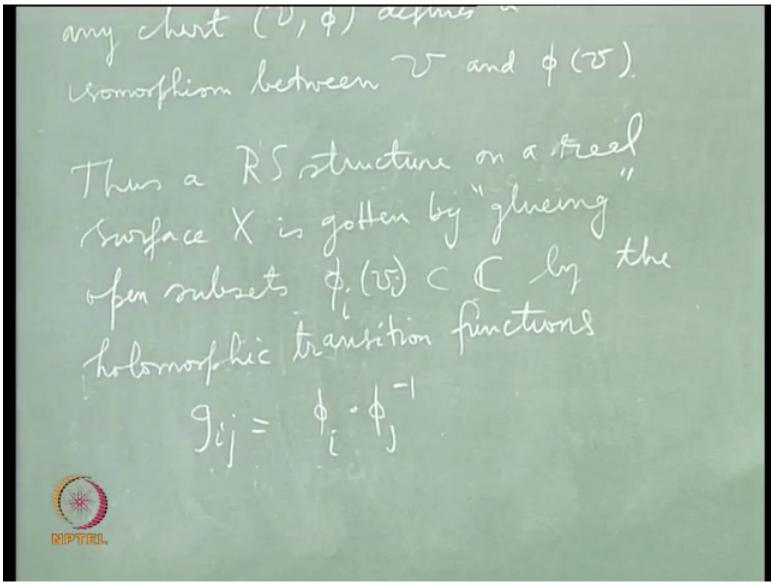

Thus a Riemann surface structure on a real surface X is got by gluing open subsets phi of U i by the holomorphic transition functions g ij which is given by phi j inverse followed by phi i. So, this is the point of view that that I would like to emphasize. So, I will stop here

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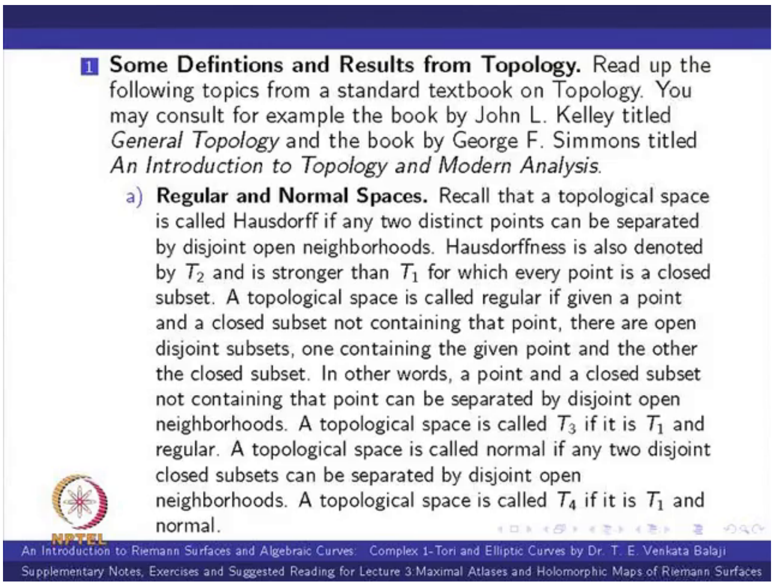

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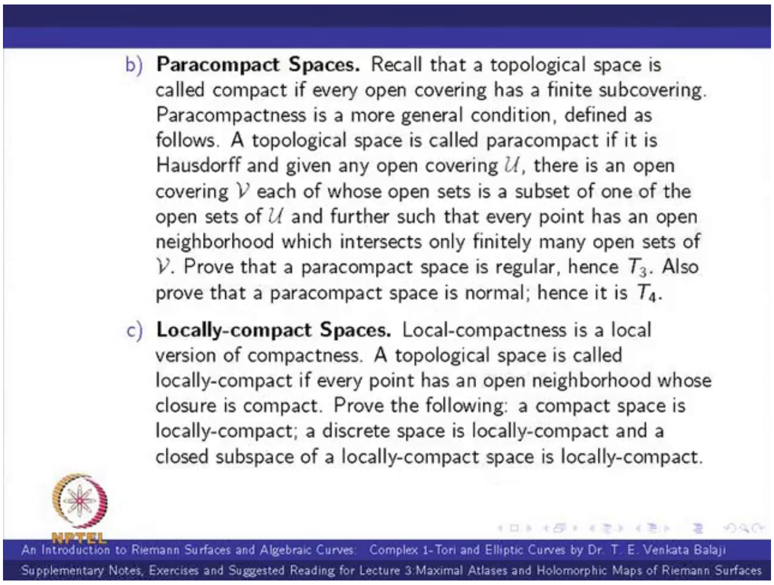

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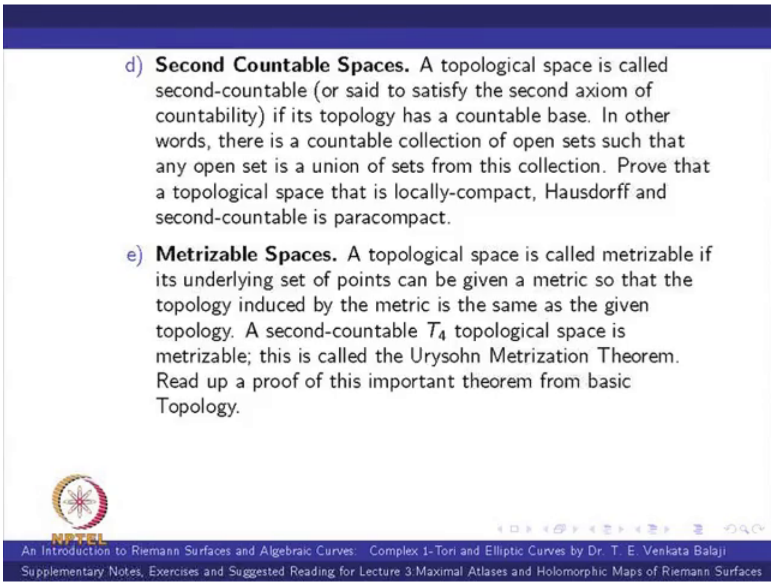

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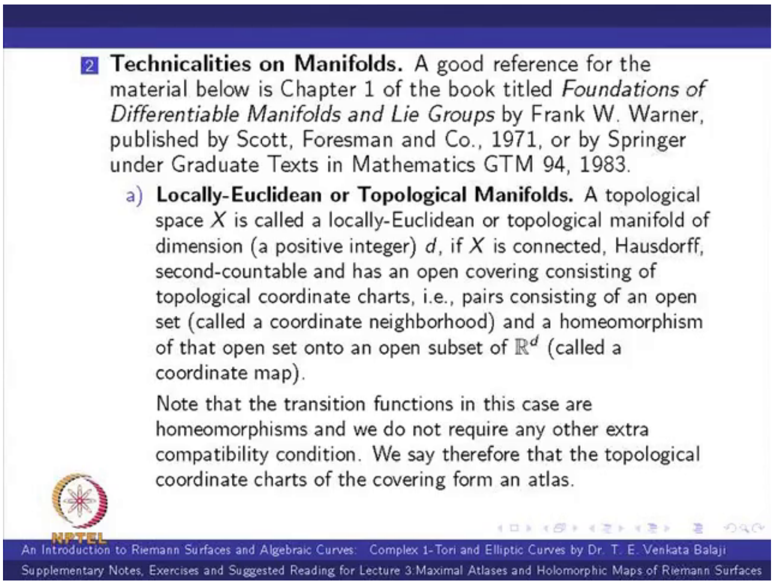

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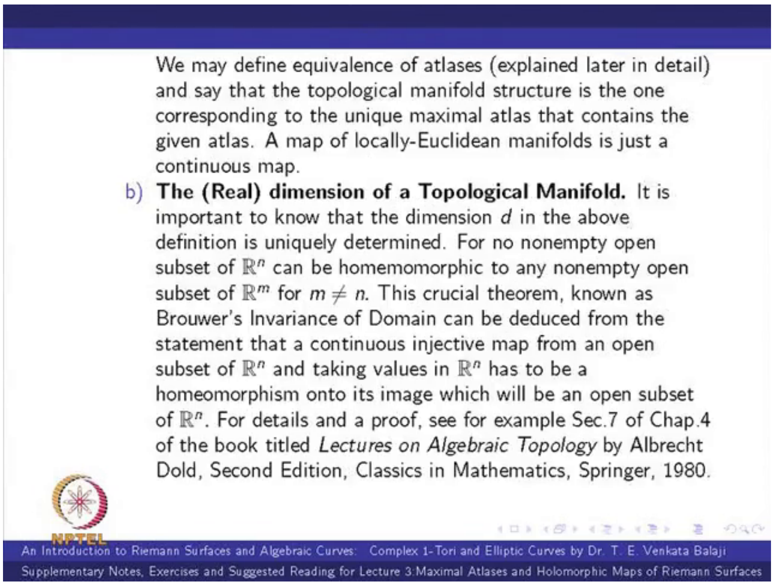

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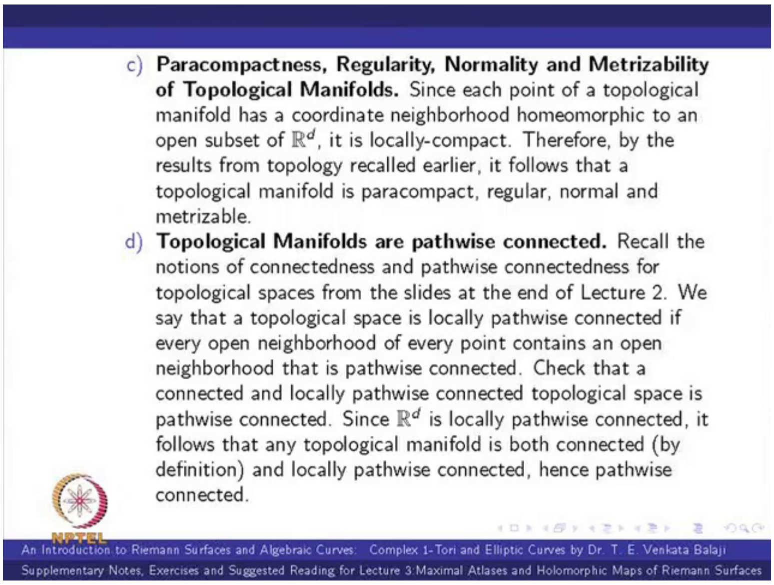

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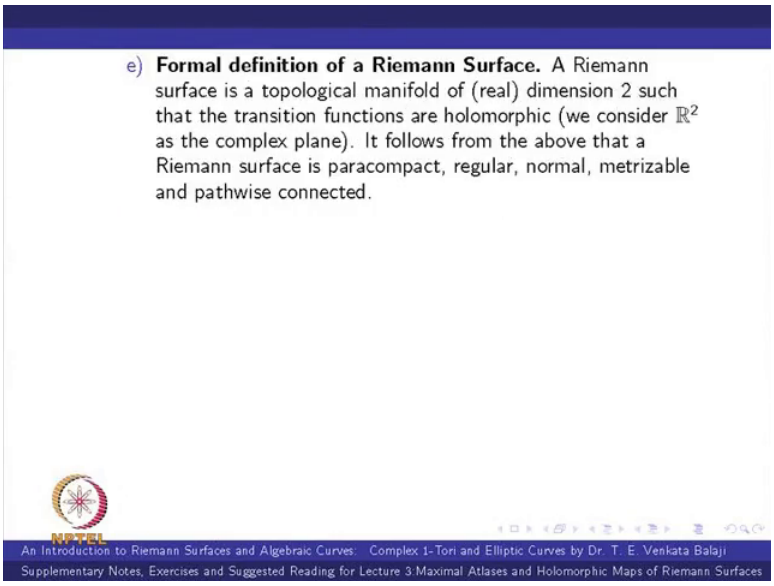

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3. **Equivalence of Atlases.** Two atlases are called equivalent iff every chart of one atlas is compatible with every chart of the other atlas (of course compatibility has to be checked only when the open sets of the charts have a nonempty intersection).
   a) Show that this is an equivalence relation on the collection of all possible atlases.
   b) Show that two atlases are equivalent iff their union is again an atlas.
4. **Existence and Uniqueness of Maximal Atlases.** Recall Zorn's "Lemma" which says that in a nonempty partially ordered set in which every chain has an upper bound, maximal elements will exist. (A chain is the same as a totally ordered subset). You may consult Serge Lang's book titled *Algebra*, Appendix 2, Section 2, either the 3rd Edition by Addison-Wesley 1993 or the Revised 3rd Edition by Springer under Graduate Texts in Mathematics GTM 211, 2002.

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Recall that Zorn's Lemma is equivalent to the Axiom of Choice as well as to the Well-Ordering Principle. Use Zorn's Lemma to prove the existence and uniqueness of a maximal atlas containing a given atlas by carrying out the following verifications.

a) Given an atlas $\mathcal{A}$, consider the collection Σ of all atlases that are equivalent to $\mathcal{A}$. Then Σ is nonempty (why?) and we put the partial order $\mathcal{B} \leq \mathcal{C}$ iff every chart of the atlas $\mathcal{B}$ is a chart of the atlas $\mathcal{C}$. Verify this is indeed a partial order.
b) If $\mathcal{B}, \mathcal{C} \in \Sigma$, show that $\mathcal{B} \cup \mathcal{C} \in \Sigma$. You may need to use the following fact: to check that a map on an open set of the complex plane is holomorphic, it is enough to check that it is holomorphic when restricted to each member of an open covering of that set. Now let $C = \{\mathcal{B}_\lambda \in \Sigma : \lambda \in \Lambda \neq \emptyset\}$ be a chain i.e., a totally ordered subset of Σ. Show that the union of all the elements of C is again an element of Σ and is an upper bound for C. Apply Zorn's Lemma to conclude that Σ has a maximal element.

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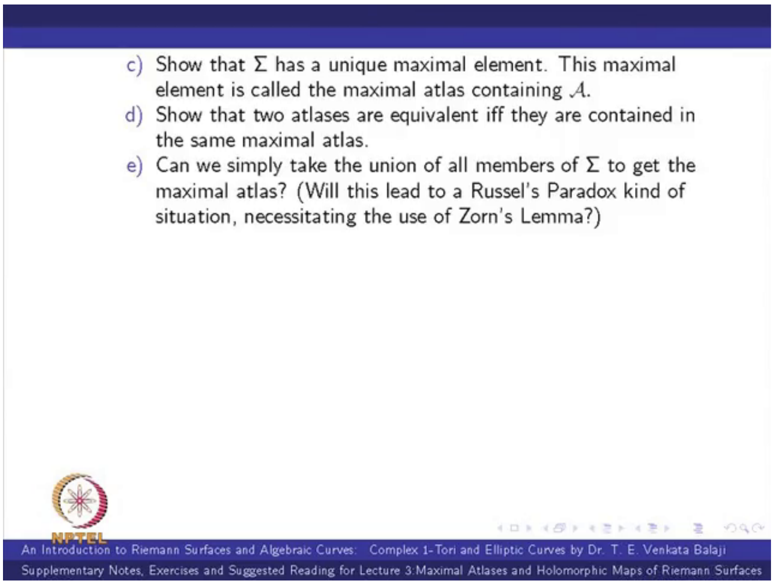

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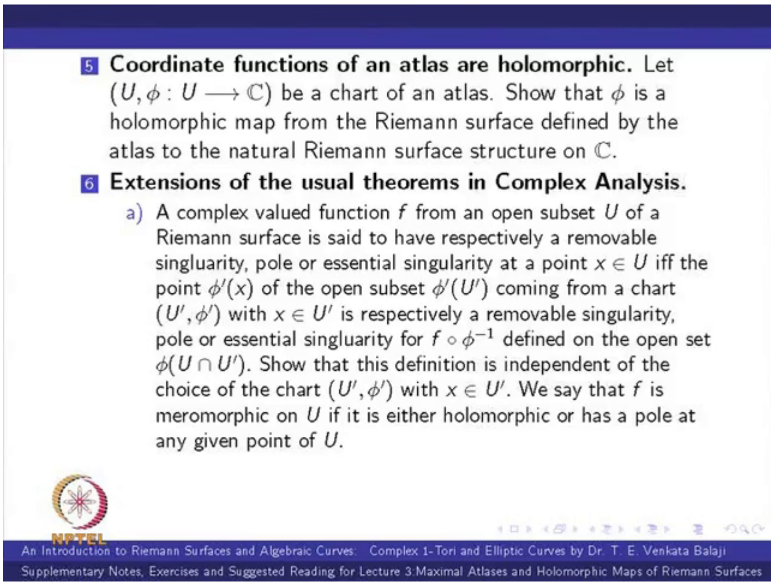

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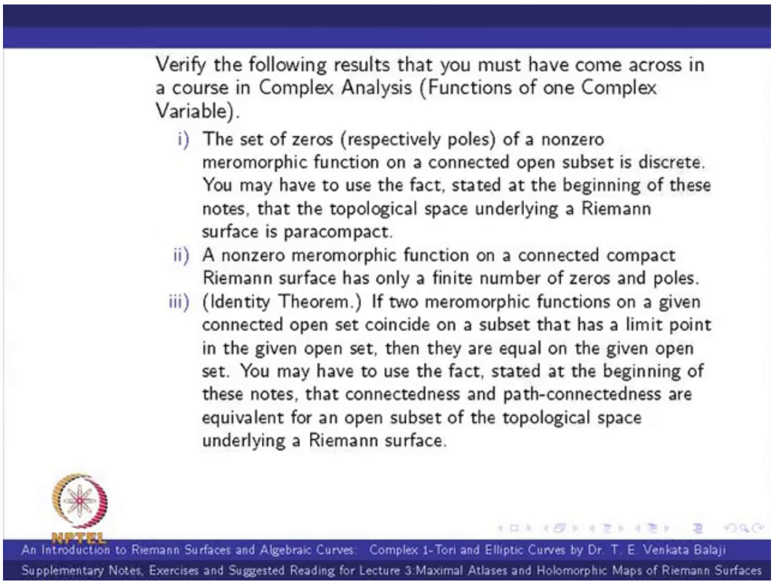

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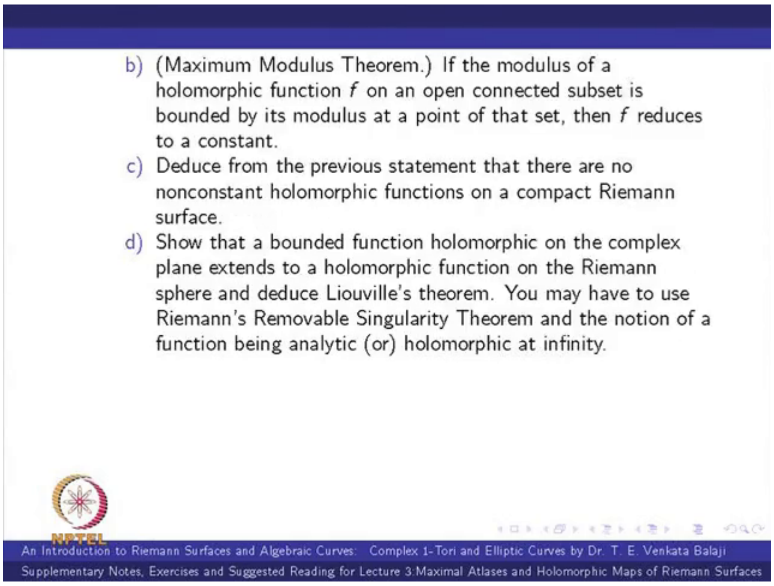

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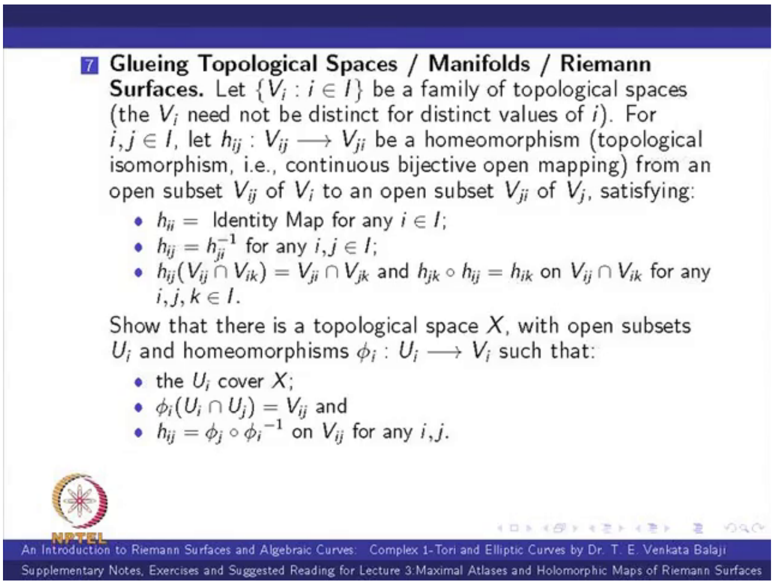

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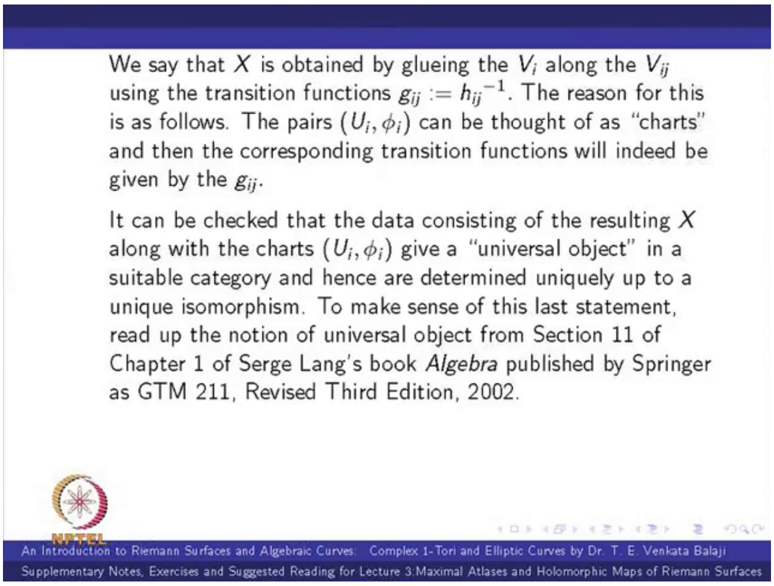

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a) **Local and Global Properties after Glueing.** Local topological properties satisfied by every V_i will continue to be satisfied by X, whereas the corresponding statement is not true for global properties. For example, if each of the V_i is locally pathwise connected or locally compact, then the same will be true of X. On the other hand, global properties like being Hausdorff, regular, paracompact, normal, metrizable, connected, second countable and so on even if satisfied by each V_i will not carry over to X. We will have to check if these hold for a given glueing. In particular we may deduce the following statements.

b) **Glueing Open Subsets of $\mathbb{R}^d$ to get a Topological Manifold.** Glueing open (even connected) subsets of $\mathbb{R}^d$ need not result in a topological manifold, so we need to check that the resulting topological space is Hausdorff, second countable and connected. If this is so, then the glueing has indeed produced a topological manifold.

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c) **Glueing Open Subsets of the Complex plane to get a Riemann surface.** We may glue open subsets of the complex plane with the additional condition that the transition functions are holomorphic. If the resulting topological space is Hausdorff, second countable and connected, the result is a Riemann surface. For now the charts (U_i, ϕ_i) are indeed local complex coordinates. This justifies the last remark of Lecture 3, which says that a Riemann surface is obtained by holomorphically glueing open subsets of the complex plane using holomorphic transition functions along smaller open subsets. In the same vein, we may say that a topological manifold is obtained by (homeomorphically or topologically) glueing open subsets of $\mathbb{R}^d$ using homeomorphic transition functions along smaller open subsets. The ubiquity and beauty of the glueing process is that it could produce spaces with new global properties. For example, suitably glueing $n+1$ copies of $\mathbb{R}^n$ (respectively $\mathbb{C}^n$) produces n-dimensional projective space which is compact, a property that none of the original spaces possessed!

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