An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1 -dimensional Tori and Elliptic Curves

Dr. Thiruvalloor Eesanaipaadi Venkata Balaji
Department of Mathematics
Indian Institute of Technology, Madras

Lecture - 28
The Unimodular Group is Kleinian

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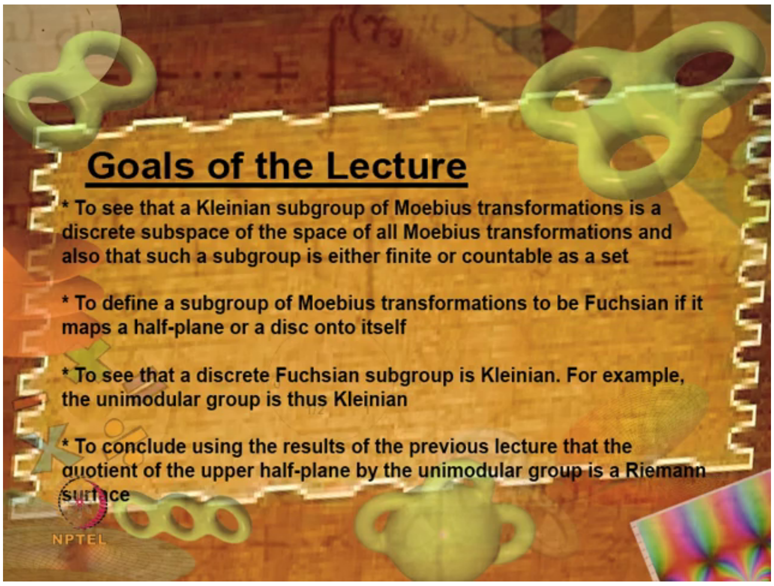

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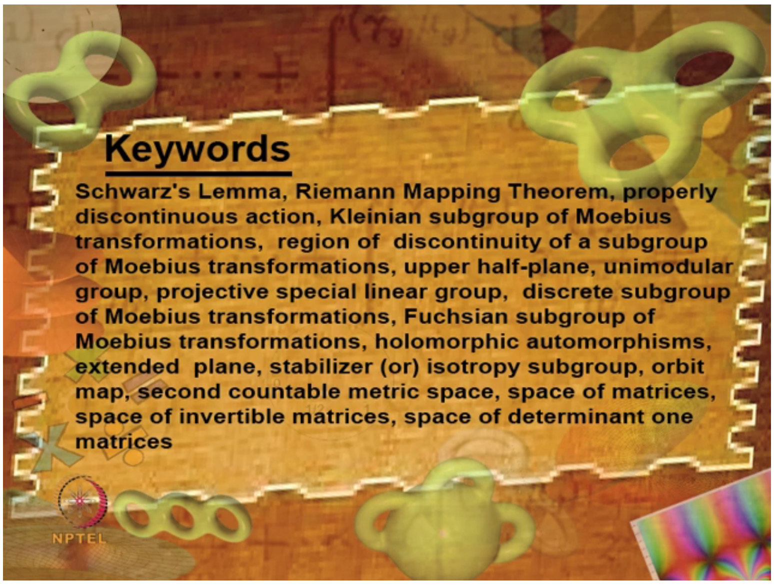

So, in the last lecture, what we saw was that if G is a sub group of Mobius transformations. That is acting properly discontinuously at least at one point; namely the reason of discontinuity of G is non-empty, in which is exactly the case when G is Kleinian group. When you seen that omega of G mod G there is the set of G orbits in omega g, is it a union of Riemann surfaces. In particularly if omega G is connected then this is the Riemann surfaces.

So, why did why were we interested in this we were interested in this because, we wanted to show that the upper half plane modulo PSL to 2 z is a Riemann surface. So, that the natural map from upper half plane to this is a holomorphic map. Now the only thing that remains to be shown is that PSL to 2 z is Kleinian group. You will have to show it is a Kleinian group. Now some work has to be done to show that it is a Kleinian group. And so, in this context will be looking at discrete subgroups, and we will also be looking at Fuchsian groups.

So, let me begin by making a remark.

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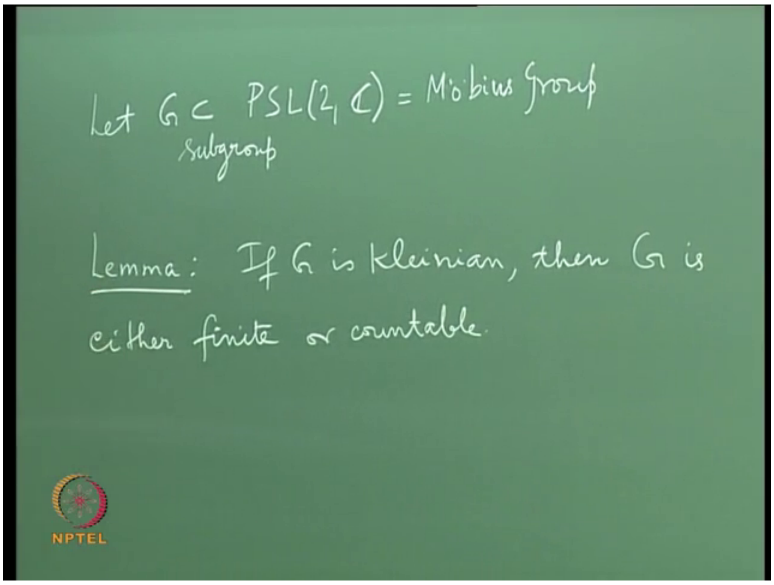

So, capital G be a subgroup of PSL 2 C, this is the Mobius group. This identified as the Mobius group. And I mean this is the group of Mobius transformations. And of course, you know these are the automorphisms, these are all the holomorphic automorphisms of the extended complex plane. Now the first thing I want to; so, here is a here is a lemma, if G is if G is Kleinian. Then G is either finite or countable.

So, actually our discussion is trying to show actually that a Kleinian group is finite or countable. And then you know if it is finite or countable, and if you could somehow deduce from that that it is discrete, which is what we are going to do after this. Then, we get that Kleinian implies discrete. And if you take PSL 2 z, I am trying to show that PSL 2 z is Kleinian. But what I know what I can immediately see about PSL 2 z that it is discrete, because it is the image of S L 2 z and S L 2 z is discrete.

So, I will have to say something about the topology on a S L 2 z and PSL 2 z which I will do, but what I must tell you is that the discussion is trying to connect the Kleinian nature with discreteness. And that is going to help us to decide when a group is Kleinian under certain special circumstances, and PSL 2 z will form will fall in this category of special circumstances. So, and of course, therefore, the only thing that will be left is to see that PSL 2 z is discrete which is anyway obvious.

So, you see so, the first lemma says that if you have a Kleinian group, then the Kleinian group as a set it is either finite or countable set. And what is the proof of this?

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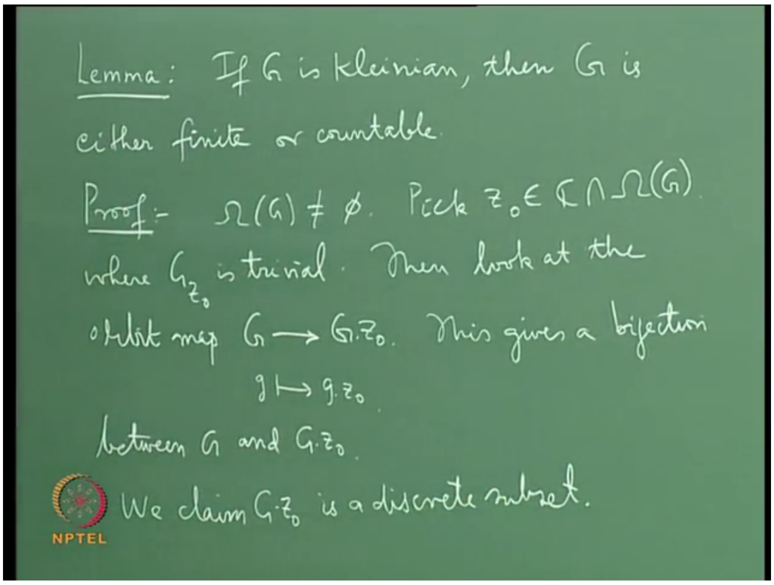

The proof of this is well omega of G is non-empty. So, it means that there is at least one point at which G is acting at least one point in the external complex plane where G is acting properly discontinuously. And well you know that that point can either be a point trivial stabilizer, or it can be a point with nontrivial, but of course, finite stabilizer.

And you know if it is a point with finite stabilizer then as we have seen the local picture it is surrounded by a neighbourhood which is full of points which have trivial stabilizer. So, in any case I can always pick a point, we have finite a point in the finite complex plane, point different from the point at infinity, where G acts freely where this stabilizer is trivial.

So, pick a point z naught belonging to c intersection omega of g, where G z naught is trivial we can do this. And then my claim is you see, then, look at the orbit map G to G dot z naught. This orbit map will be it will be a bijection. Because so, this is the map that sends G to G dot z naught. And you know because there is no nontrivial element of G which leaves is z naught fixed, which is which is what it means when you say that the stabilizer is trivial. This map is a injective and by definitions surjective. And therefore, you get a bijection between g and g dot z naught.

So, this gives a bijection between G and G dot z naught the orbits. So, G dot z naught is all those elements of this form small g dot z naught where small g is in capital G. So, I want to show that capital G is finite or countable. Therefore, it is enough to show that this orbit is finite or countable. Now what I want to say is that, this is obvious, it is because you see take the take the orbit take the orbit here. Then my claim is that this orbit is a discrete subset.

So, we claim G dot z naught is a discrete subset. It is a discrete subset of course of C. In fact, if you want of even of omega of c. It is a discrete subset. And why is it a discrete subset, because if it is not discrete subset it will have an accumulation point. And if it had an accumulation point, then what will happen is that you will find a sequence of points in the orbit coming close to that accumulation point. And then; that means, you will have you can find points in the orbit which are as close as you want different points in the orbit, that are as close as you want and that cannot happen because of the fact that the stabilizer is trivial and every element small g of capital G, this there is a special neighbourhood of a z naught, which displaces which is completely displaced away from itself by any nontrivial element of capital G that will get contradicted. So, that should tell you that G dot z naught, cannot have an accumulation point. So, it will be a discrete subset.

So, if you want let me write down the argument for let z 1 be an accumulation point of the orbit.

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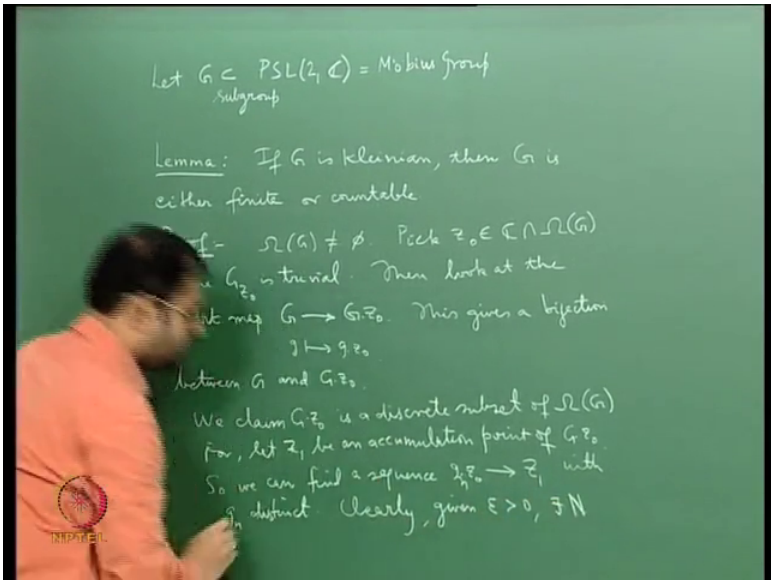

And so, what is going to happen is so, we can find a sequence. Well, let me say g n of z naught tending to z 1. You can find a sequence of points in the orbit that tends to z 1 with of course, with g n distinct, because z 1 is an accumulation point. You can after all pick a system of neighbourhoods of z 1, and you can rad decrease the radius of the system of neighbourhoods, and in each in each one you can find a point of the orbit and therefore, you get here you know you get a countable sequence like this of points in the orbit that tends to z 1.

Now, this is the con this will give you a contradiction. Why because, see this sequence is Cauchy, after all this sequence converges to z 1 therefore, this sequence is Cauchy. So, clearly given epsilon greater than 0, there exists an n such that so, let me write, let me continue here such that well m greater than n greater than n implies that the distance between g n of z naught and g m of z naught can be made lesser than epsilon.

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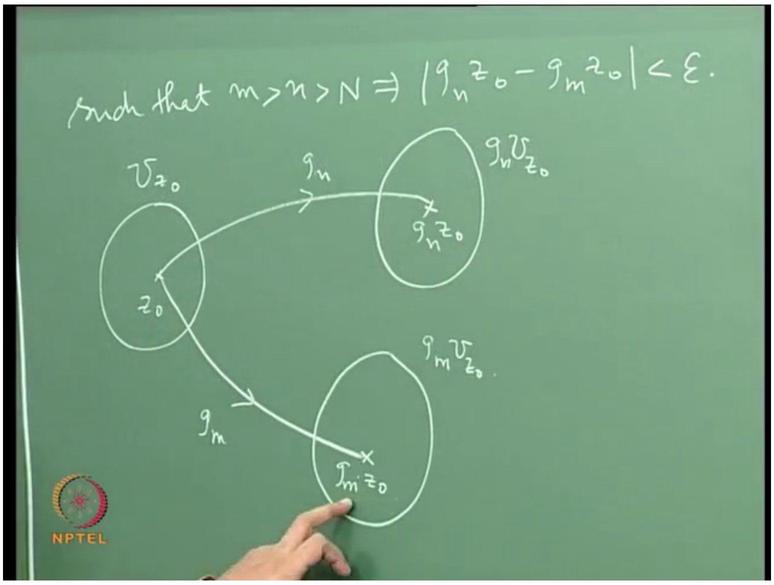

This is just the fact that the sequences is Cauchy, because it is a convergence sequence. But then you see now look at the point z naught.

Now, this point z naught well that there is a neighbourhood, u sub z naught which is the neighbourhood that satisfies the condition for proper discontinuity of the action of G at z naught. What is a condition? The condition is well if you move it if you take the image of this neighbourhood by any nontrivial element of G, you end up with translate with does naught intersect this neighbourhood. So, if I move this neighbourhood by let us say, g n then I will get I will get this neighbourhood here I will get I will get a translate of that which is g n u z naught. And of course, the point z naught will move to this point which is g n of z naught. And if I take a well if I take a different g m, g m different from g n which is which I have already taken here.

Then well I will get another translate which will be g m of u z naught. And of course, z naught we move to the point g m dot z naught. And these 2 neighbourhoods cannot intersect that is because g n and g m are distinct. Because if these 2 neighbourhoods intersect, then you would have then you would contradict, then you would contradict the fact the this is this neighbourhood is completely displaced from itself by every nontrivial element of G, see if you want I can write that down g m u z naught intersection, suppose g m g n u z naught if this intersection is non-empty.

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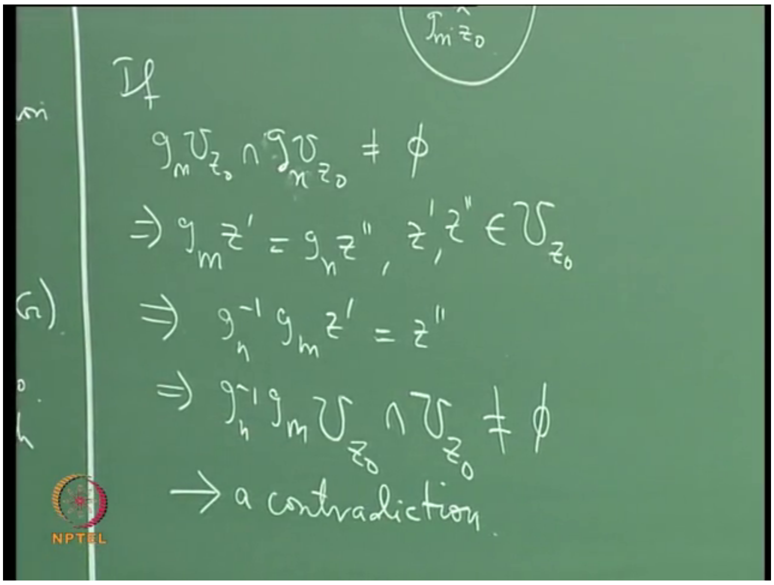

So, what it will imply is that it will imply that g m dot some z prime is equal to g n dot e z double prime, where z prime and z double prime are in u z naught is what it will imply.

But then you see what this will also imply it will imply that g n inverse g m of z prime will be z double prime, because after all I can operate by g n inverse on this side being a group, but then this will imply that g n inverse g m of u z naught intersection u z naught is non-empty, because these contains z double prime; this contains z double prime, which is also point here. And a this is this is the contradiction, a contradiction. Why this is a contradiction? Because g n inverse g m is not the identity, g n inverse g m is a nontrivial, element of G and every nontrivial element of G is supposed to push displace u z naught completely away from z naught. So, it is a contradiction.

Therefore, the diagram is as I have shown, these 2 do not intersect. But if as and this holds for any n and m. So, if you let n and m to tend to infinity, the distance between g n z naught and g m z naught cannot be made arbitrarily small which is what this condition says. So, it is a contradiction. So, well let me write it here clearly.

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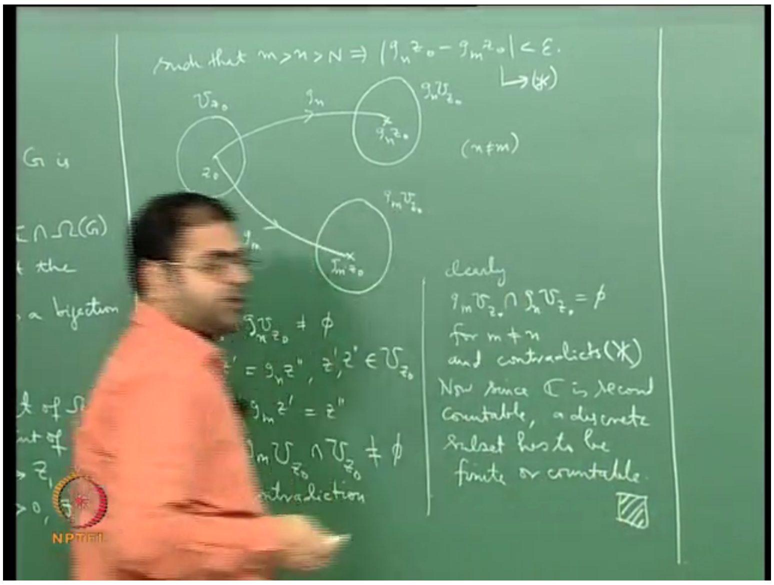

G m of u z naught intersection g n of u sub z naught is empty for m naught equal to n. Of course, I am I am assuming m is not equal to n. This intersection is empty, and contradicts if you want this condition well if I call this condition as star it contradicts star obviously. So, the moral of the story is that G dot z naught is a discrete subset of omega G. Now I use the following fact. I use the following fact, that if you are looking at if you are looking at second countable metric space, then in a second countable metric space; A discrete subset this always finite or countable. In particular omega G is a subset is a subset of c, which is or 2 topologically; it is of courses a second countable matric space. And discrete subset is therefore, finite or countable.

So, the fact that I am using is that in a matric space which second countable. A discrete subset is you know, going to be finite or countable. The answer to the proof of that statement is very simple, because the matrix space is second countable what you can do is you can label all the neighbourhoods the you can find a basis for the topology of that space, by set which can be labelled by let us say natural numbers. And given a discrete set, I can find I can find the least such index among this collection that is going to contain only that point. And in this way, I can make this subset of the index set of that collection. And that will give and that along with the fact that subset of a countable set is countable or finite will tell you that G dot z naught is either finite or countable.

So, I am using a second countability here right. And of course, you know all Euclidian spaces are second countable, because for example, if you if are taking complex numbers you can take you can get a countable basis by looking at all the you look you take the collection of rational point with the rational coordinates, and for each such point you take all the balls I mean open disks with rational radii. That if you take the union of all that that is a union of it a countable union of countable sets and which is again countable. So, let me write that down. Now since the c is second countable second countable a discrete set a discrete subset has to be finite or countable. So, that is end of the proof of this. Because I have shown that the orbit is finite or countable, and the orbit it is bijective to G and is bijective to G because I have chosen a point with trivial stabilizer.

So, Kleinian group is finite or countable. Now I am, I next want to say I want to say more I want to say in fact, that a Kleinian group is actually discrete. I want to say that Kleinian group is actually discrete, but then to say Kleinian group is discrete, I must have some top some kind of topological structure on this. So, I just want to explain what that topological structure is very quickly. You see we have M 2 C.

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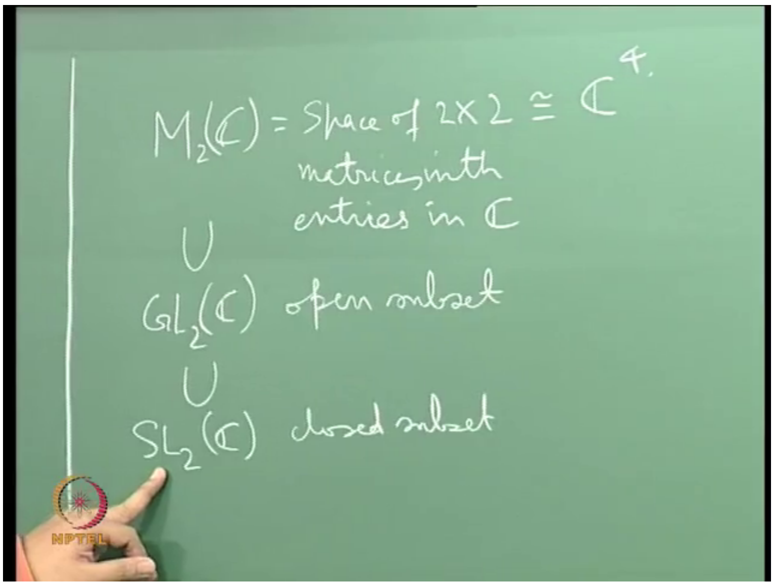

This is the, this is you see this is just this is just space of 2 by 2 matrices with entries in complex numbers. And this can be well this can be identified with c to the 4, because have 4 entries. So, you can identify with, identify it with of the 4-dimensional complex space right. And well and in M 2 C, if you take G l 2 C if you take G l 2 C then this is the this is an open subset. See, M 2 C can be given a topologic topological structure even a metric space structure, because c 4 has all those structure.

So, you can make this in to a topological space, you can make it into a metric space. But if you take a matrices a b c d with entries a b c d, you just associate it to the 4 tuple a b c d. So, this is nothing but c 4 in in a disguised way. And what is G l 2 C? These are all those points these are all those matrices at are invertible; that means, this is the this is the set of points where the determinant function is nonzero, but you see the determinant function is a polynomial function, it is a polynomial function of the coordinates. So, it is continuous and the set of points where continuous function does not vanish is an open set.

So, this is an open subset open subset, this is an open subset and these also a inherits sub space topology and a metric and so on. And in this I can further go down and look at S L 2 C S L 2 C is a closed subset. It is a closed subset here and as in fact, it is a closed subset there itself, because these are all those matrices with determinant one matrices with determinant one is you are looking at this 0es of a continuous function. Therefore, this is a closed set there itself and of course, again this also inherits a topology and a metric space structure and so on so forth and now how do I get PSL 2 C what I do is I just go I just go mod the plus or minus identity.

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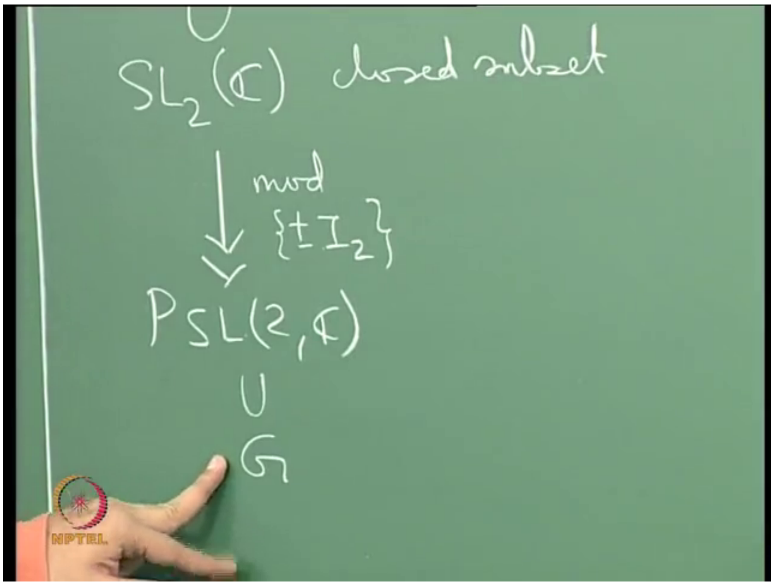

So, I capital I sub 2 is a 2 by 2 identity matrix, and this plus or minus identity is a of course, a normal sub group of this group if you want. And mind you these 2 are groups also and multiplication and the and the multiplication turns out to be also continuous. So, well if I go mod plus or minus I 2 I get PSL 2 C. So, well sometimes I write S L subscript 2 C sometimes you can also write S L 2 comma c that should not cause any confusion.

So, if you go mod plus or minus identity then you get this PSL 2 C, and you see what you are doing here is you are going modulo a subgroup a normal subgroup. If you can actually check that this is a covering this is a sheeted covering. Because plus or minus identity is going act freely on this. And this is a covering and therefore, the moral of the story is that this will also inherit you know, all the properties of this. So, it will be a metric space, and it will be you know how stop and so on so forth.

So, the point is that when you look at Mobius transformations, Mobius transformations are elements here. You can always look you can talk about convergence of the sequences of Mobius transformations. As points here, and you can also talk about you can you can also talk about a subset. So, this has a topology. So, if G if I look at a group G a sub of PSL 2 C it is a subgroup of this, then I can talk about where I do not even need a subgroup I can even have a subset here. And I can talk about when the subset is a discrete; because now this has a topology and I know what discrete means it even has a metric.

So, well so, given this background, it makes sense to talk about a subset of PSL 2 C being discrete and more so for a subgroup. So, we make a definition that we say a sub subgroup of PSL 2 C that is a group of Mobius transformations is discrete. If it is discrete as a in the in the topological space underline PSL 2 C we make that definition.

So, let me let me write that down definition.

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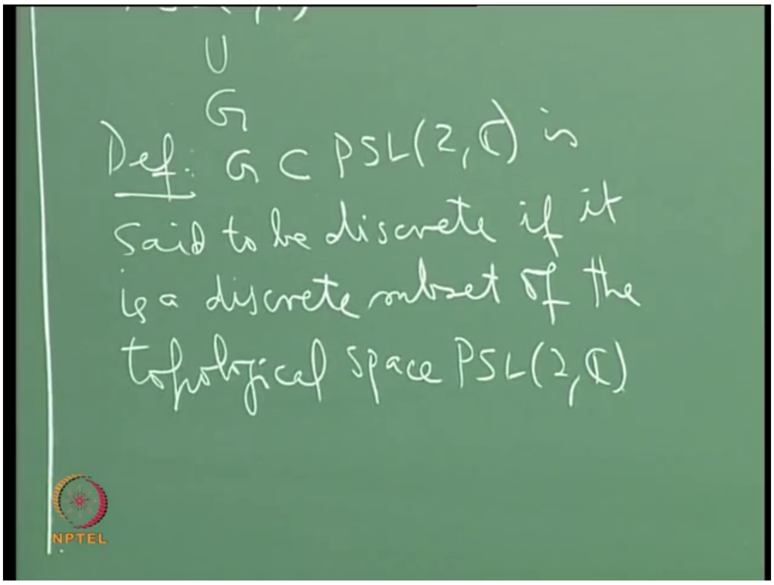

G in PSL 2 C is said to be discrete, if it is a discrete subset of the topological space of the topological space PSL 2 C. So, in particular I can define what a discrete subgroup of Mobius transformations is. It is a subset which is discrete and is also a subgroup.

Now, having said this, what am I going to see next I am going to say that, suppose z is a Kleinian group, then as a subset of the topolo of the of the topological space underline PSL 2 C, I claim that it is a discrete subset. So, here is a well lemma.

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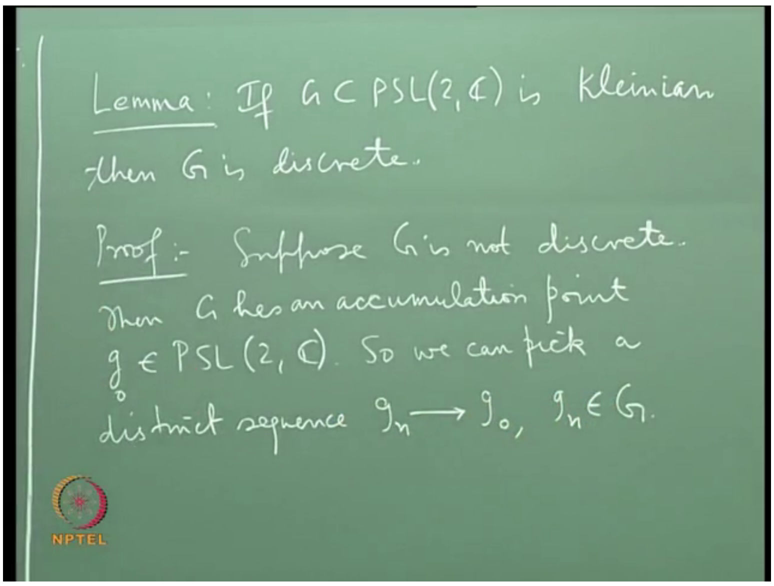

If G in PSL 2 C is Kleinian, then G is discrete. So, this is stronger than that lemma, this is stronger than that lemma. Because if I knew G is already discrete then you can believe that PSL 2 C is second countable. Because for PSL 2 C everything comes from above, and c 4 is of course, second countable. So, PSL 2 C being second countable, you know any discrete any discrete subset is of course, finite or countable.

So, this will be implied by this. So, this is the strengthening of that lemma. So, how will how will one prove this? Proof is well; it is essentially the same kind of argument. I assume that G has an accumulation point. And I prove that I get a contradiction to the action of G being properly discontinuous. So, I assume G is Kleinian I assume G is Kleinian. Therefore, the actions of G, there are points which at which G acts properly discontinuously I can get a contradiction to that if G has an accumulation point. So, that is the proof. So, let me look at it for a moment.

So, well suppose G is not discrete, suppose G is not discrete. Then G has an accumulation point G small g let me call it as G naught if you want G naught in PSL 2 C. G is an accumulation point in of course, I am saying it is a discrete subset of this. So, this is the ambient space this is the biggest space. And if I assume it is not discrete, then I should have accumulation point there. So, what it means is that. So, we can pick a distinct sequence g n which is which goes to G naught where g n are all from your g. So, I can pick up distinct sequence.

Now, you see again if g n tends to G naught, if z is any point, then in particular if I take a point which is a point in omega of G, a point where G acts properly discontinuously; then of course, if g n tends to G naught it is not hard to verify the g n z tends to G naught z. Now g n z tends to G naught z will tell you that g n z is going to be a Cauchy sequence. But then if g n z is a Cauchy sequence, then I will get a contradiction to the action of G being properly discontinuous at z. The argument is very similar to this. So, let me write that down. So, that will tell you therefore, that this cannot happen.

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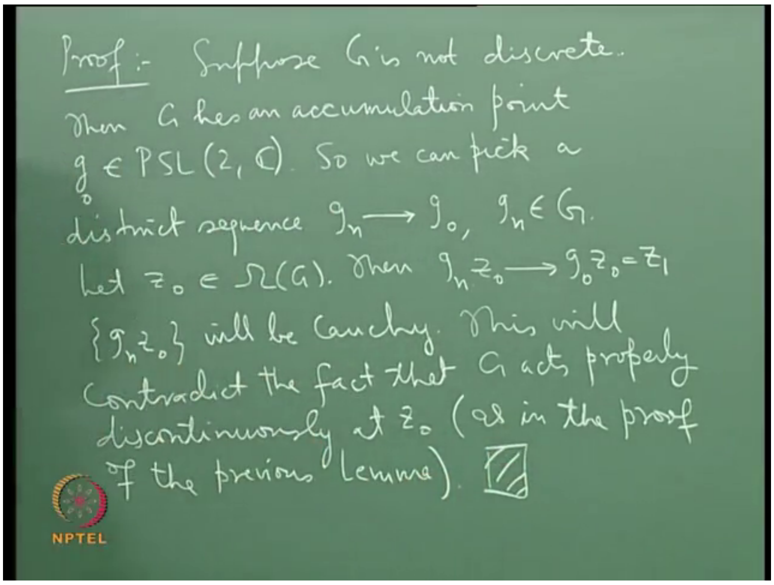

So, let z naught belong to omega of G, then well g n z naught tends to G naught of z naught, which if you want you call it as z 1 and g n of will be Cauchy. And we are in the same situation as we were here, you can of course, if you want you can even take well to be on the safer side take z naught to be a point in the finite complex plane. And if you want even take z naught to be a point to trivial stabilizer, just like we did there you can do that and g n z naught will be Cauchy. And this will contradict the fact that G acts properly discontinuously at z naught. So, as in the previous lemma as in the proof of the of the previous lemma.

So, that is the proof of this statement right. So, the only thing that has to be checked is that if g n tends to G naught in the topology here in PSL 2 C you have to check that g n of z tends to G of G naught of z, for any finite complex numbers z. That should not be very difficult to check well. So now, this lemma tells me that a Kleinian group is discrete. Now I am looking for a situation, when Kleinian and a discrete group is Kleinian. I want a situation when a discrete group is Kleinian, and once I get a condition for that if I if I can show that PSL 2 C satisfies that condition I am in good shape because then I know that PSL 2 z is Kleinian which is what I want. So, that u mod PSL 2 z will become a Riemann surfaces.

Now, unfortunately a discrete group need not be Kleinian. There is a that is a standard counter example to this; so a remark or well a warning.

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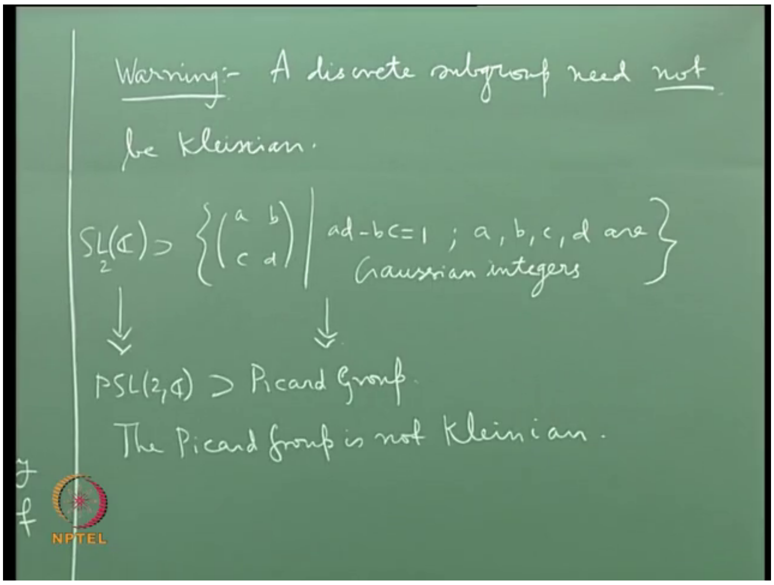

A discrete group, subgroup need not be Kleinian, a discrete subgroup need not be Kleinian. What is example? You just look at what is called that picard group. So, you see you take the set of all matrices a b c d, such that a d minus b c is 1. So, this is of course, going to be in S L 2 C these are all matrices with determinant one. So, it is going to be an S L 2 C, and I am going to put the additional condition that a b c d are Gaussian integers. I am going to put the condition that a b and c d are basic there are complex numbers, whose real and imaginary parts are the usual integers.

So, I will put this extra condition a b c d are Gaussian integers. That is the real and imaginary parts are usual integers. The fact is that you see it is very clear that it is discrete. Because the entries are all the real and imaginary parts the entries are only integers. It is very, very clear that this is discrete alright. In fact, if I put the condition that a b c d are Gaussian integers, in and look at the corresponding subset in M 2. That itself will be discrete subset of M 2.

So, and this is a subset of this is just the intersection of that subset with this. So, it continues to be discrete right, but the problem, but the big deal is that you can show it is an exercise, and I will not say it is an easy exercise it is it demands a little bit of work. You can show that this group is not Kleinian, this group is not Kleinian. So, in fact, the way it is done is that you can show that this group, the orbits of this group you can show you can find orbits of this group have accumulation points.

So, that is how that is how the proof course, so, it is an exercise also although it is a slightly harder exercise. So, let me write this let me make it as a remark. So, the image of this so, I have this quotient which is PSL 2 C. And the quotient the image here is called the picard group. This is called the picard group. It is all these determinant one 2 by 2 matrices is with entries Gaussian integers up to plus or minus 1 up to sign of course, going mod go going to be PSL 2 C is just going mod sign plus or minus. So, this the picard group is not is not Kleinian the picard group is not Kleinian.

So, the problem is therefore, Kleinian and the notion of discreteness in Kleinian nature kleinianness if you want they are not the same they are not to same, but here comes something that g n turns out to be good and god given. If you put the additional condition that the group, preserves a disk or an upper half plane; such groups are called Fuchsian groups a subgroup of Mobius transformations that preserve a half plain or a disk. They are they are special they are called Fuchsian groups. And the beautiful thing is for Fuchsian groups there is no difference between discreteness and kleinianness, that is the theorem we are going to prove. So, believe that theorems then you see immediately that PSL 2 z is Kleinian. Because PSL 2 z is discrete, and PSL 2 z leaves the upper half plane fixed. So, PSL 2 z is a discrete Fuchsian group, but then it is Kleinian therefore, PSL 2 z is Kleinian.

So, let me make that definition here definition a subgroup G of Mobius transformations are called Fuchsian if it leaves invariant or preserves a disk or a half plane.

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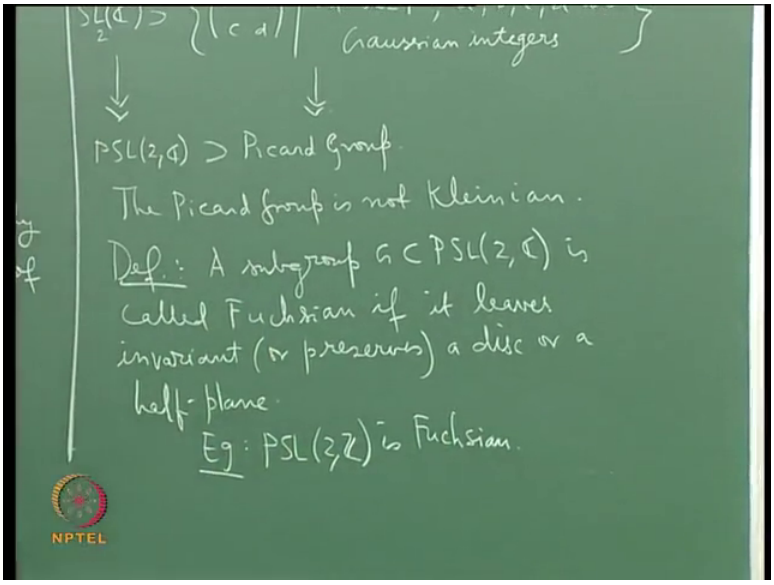

So, standard example is PSL 2 C PSL 2 z is Fuchsian. Because you know it is a subgroup of PSL 2 r and PSL 2 r is precisely the all the Mobius transformations that leave the upper half plane fixed. So, this leaves the upper half plane fixed. This leaves the upper half plane fixed and therefore, it is Fuchsian.

So now let me say that theorem which is going to be a great help to us. So, here is the theorem.

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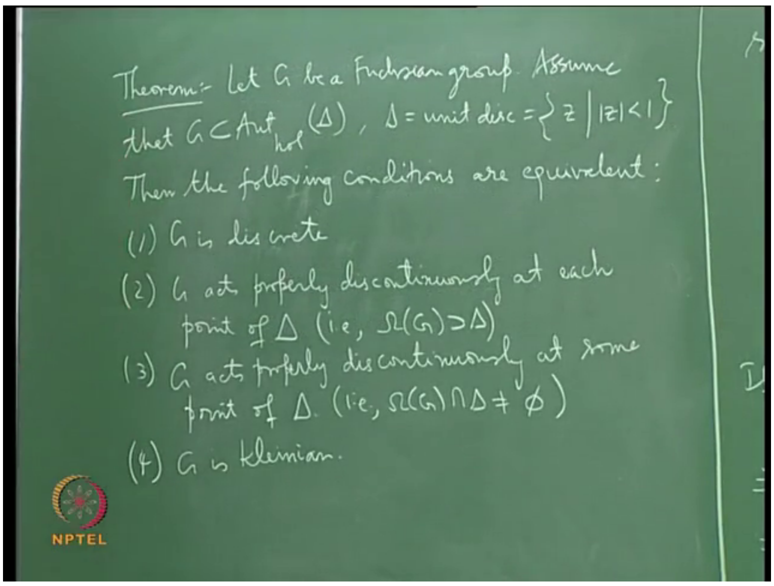

Let G be a Fuchsian group. Let G be a Fuchsian group. Assume that G is subgroup of the holomorphic automorphisms of the unit disk. Delta is equal to unit disk is a set of all z is that mod z less than 1. So, then the following are equivalent condition conditions are equivalent. Number 1 G is discrete. Number 2 G acts properly discontinuously at each point, each point of unit disk. That is another way of saying it is that omega of G contains delta. The third condition G acts properly discontinuously at one point of delta, at least at one point of delta. Seemingly weaker condition, at some point of delta; this is the condition that omega of G intersection delta is non-empty. There is at least one point of delta which is in omega of G. And the forth condition is G is Kleinian.

So, in the in the looking at the statement of this theorem I want at the outside make a few remarks see if G is a Fuchsian group, it will leave some disk or half plane invariant. Now if it is, but any disk or half plane can be mapped on to the unit disk; so you can replace G by a conjugate by a conjugate with that map to get an automorphism, you to get a subgroup of automorphisms of the unit disk. And replacing by the subgroup by conjugate subgroup is not going to affect any of these properties, discreteness. You know, kleinianness, you know the action been properly discontinuously so on and so forth.

So, that is why without loss of generality I have assumed that G is a subgroup of holomorphic automorphism of for the unit disk. And I am proving the statement proving the theorem for that case. But what I want to saying that there is no loss of generality. Well, the other thing is what I want to tell you is 2 implies 3 is trivial, because 3 is weaker than 2 3 implies 4 is the definition of Kleinian group. A group is Kleinian if of course, for G to be Kleinian all I need is omega of G is non-empty. So, if in particular if omega G is intersection delta is non-empty then omega G is non-empty therefore, 3 implies 4 is also directed by definition 4 implies one was the was the lemma that we proved.

So, the nontrivial part is actually one implies 2, which is what we will focus on.

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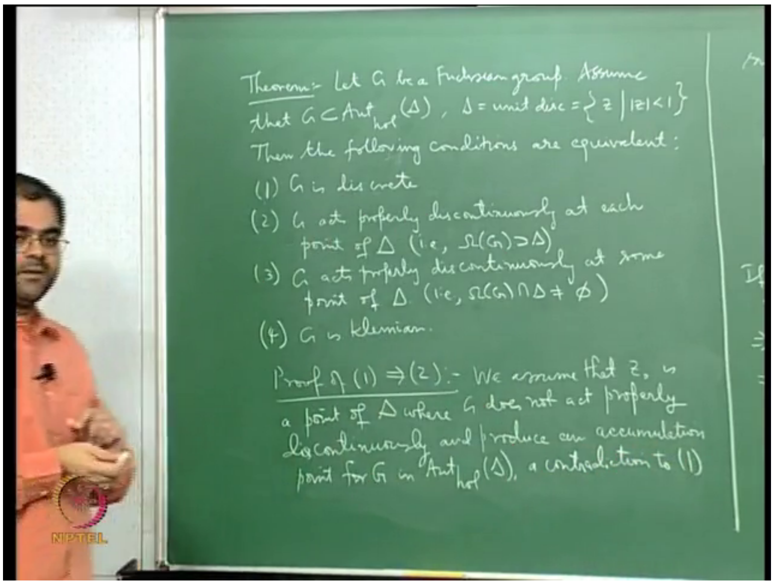

So, proof of one implies 2. So, I am just going to prove that I am assuming that G is discrete. And I am going to prove that G is going to act properly discontinuously at each point of delta. Mind you the way I am going to prove it is by contradiction. I am going to assume that there is a point of delta where G does not act properly discontinuously, and using that point, and you know special knowledge about the automorphism group how these automorphism group looks like and Schwarz's lemma and going to cook up an accumulation point for G which will contradict one. So, it is a very clever proof, but not all that complicated.

So, let me do that assume we assume that z naught is a point in delta which is a point in delta minus omega of G. Rather I should write no there, isn't enough space here. Let me write it words we assume that z naught is a point of delta where G does not act properly discontinuously, and produce an accumulation point for G in automorphism group of delta a contradiction to 1.

So, what am I going to do? I am going to take a point in the unit disk, where G does not act properly discontinuously. And using that point I am going to cook up an accumulation point for G in here, and that will contradict the fact that G is discrete. So, well let us let us look at how this is done. So, let me make a few remarks to begin with.

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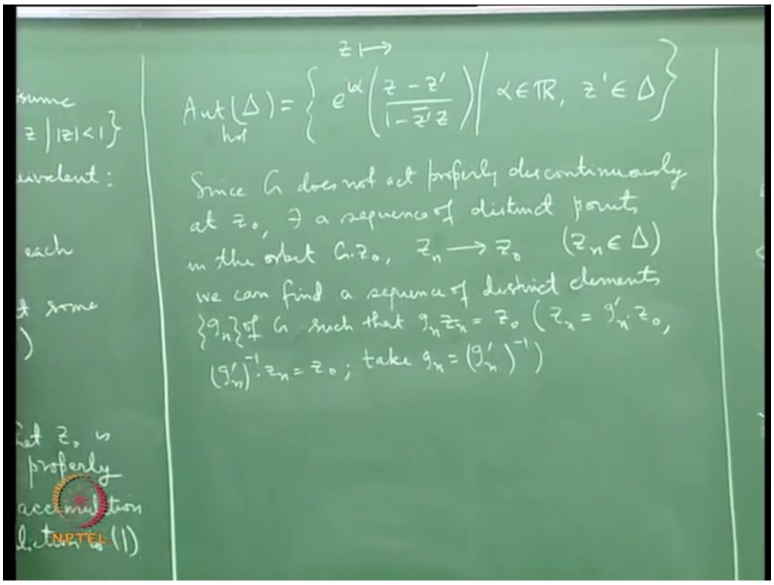

See the automorphism group of delta the group of holomorphic automorphism delta what does it look like, that is something that is probably an exercise you would have done in first course in complex analysis. Let me recall it is the set of all bilinear transformations of the form e power i alpha times well I should say z goes to e power i z e power i i alpha times z minus z prime by 1 minus z prime bar z where alpha is a real number. And z prime is an element of delta. You see, z prime will go to 0, under this map. This is the Mobius transformation. This is the Mobius transformations. And it is precisely these and z prime is the point in the unit disk. And these are precisely all possible automorphisms of the unit. Disk this is this is a this is an exercise in a first course in complex analysis you should do it if you have not done it. And it is not difficult to do.

Now, what I am going to do is that well, you see since z naught is a point at which G does not act properly discontinuously. What it means is that you cannot find that you cannot find any neighbourhood of z naught, which is completely displaced from itself by you know all bit of finite number of elements of G. So, I cannot find such a neighbourhood alright well. So, from that what I can do is I can cook up a sequence of distinct points in the orbit of z naught which you know converts to z naught.

So, since G does not act properly discontinuously at z naught, there exists a sequence of distinct points in the orbit G dot z naught z n tending to z naught. And of course, these z ns are all in the unit disk. Mind you, z naught is also in the unit disk. I get this because G is not acting properly discontinuously a at z naught. And then we can we can find a sequence of distinct elements of g n of G such that g n takes the z n to z naught you can you can find the sequence of distinct elements which take g n which take the each of these z ns respectively to z naught.

This is possible because you know after all g n is z n is well you see it is in the orbit of G z naught. So, it is some g n prime dot z naught. So, that tells me that g n prime inverse dot z n is z naught and I have to take g n to be equal to well g n prime inverse. So, this is this so, this is possible. And well the first thing that one wants to do is cook up a sequence of elements here you have seen these z ns the z ns and z not.

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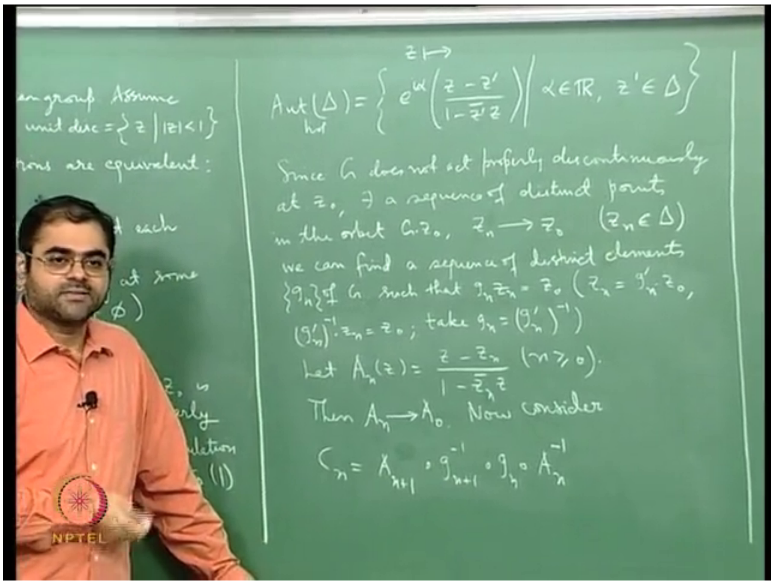

So well, put let A n of z be just you know z minus is z n by 1 minus z n bar z take this. Then of course, you know for n of course, n greater than or equal to 0 of course, I can include case n equal to 0. Then then A n of then A n of course, tends to you know a naught, as n tends to infinity z n tends to z naught. So, A n of z tends to a naught.

Now, consider C n to be you know, you take A n plus 1 inverse composition well g n plus 1 composition g n composition I guess A n may be I will have to put I have to put inverse here, and I remove the inverse here. So, let me check this it easy to that is right. The reason why you are doing this is very there is an obvious reason. I want C n to fix the origin I mean this you adjust this in such a way that C n fixes the origin. And you know, the thing in between these 2 composition of these 2 they are all in G that is what I want.

So, you see it is it is quite easy to see that C n. Of course, C n is a composition of automorphisms of delta.

And therefore, it is certainly an automorphism of delta there is no doubt about that. And you see if you calculate you see each A n if you take C n of 0 what I will get this I will get A n plus 1 of g n plus 1 inverse of g n on well A n inverse of 0.

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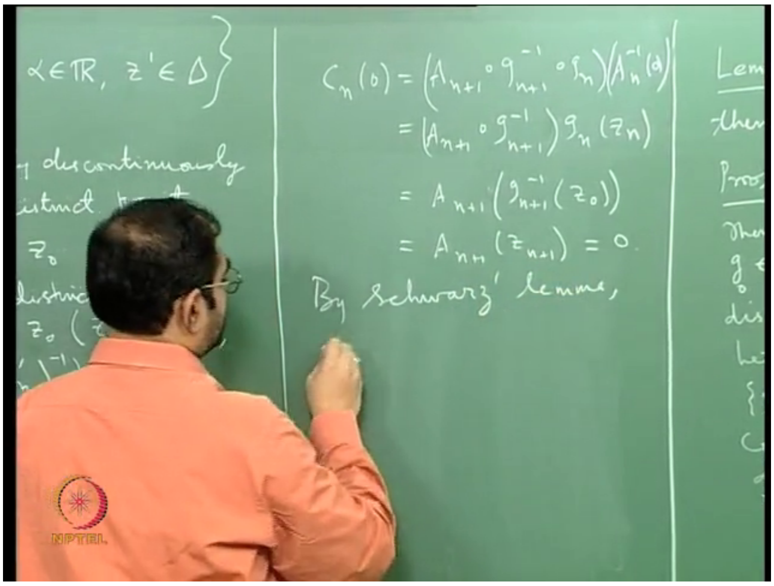

Well, let me let me write that. And you see A n takes z n to 0, and it is a you know (Refer Time: 57:07) Mobius transformations are bijective.

So, A n takes z n to 0. So, n A n inverse of 0 is z n. So, what you will get is this will this will become A n plus 1, composition g n plus 1 inverse composition well acting on g n of z n. But g n of z n I have chosen it to be z naught alright. So, this becomes A n plus 1 composition g n plus 1 inverse I mean acting on g n plus 1 inverse of z naught. But you see g n plus 1 inverse of z naught is z n plus 1. So, what I will get is I will get A n plus 1 of z n plus 1 and that is 0. So, the moral of the story is I am getting an automorphism of the unit disk, which fixes the origin. Or now I can use Schwarz lemma and say that this is the rotation. I mean that is the whole point of cooking up this.

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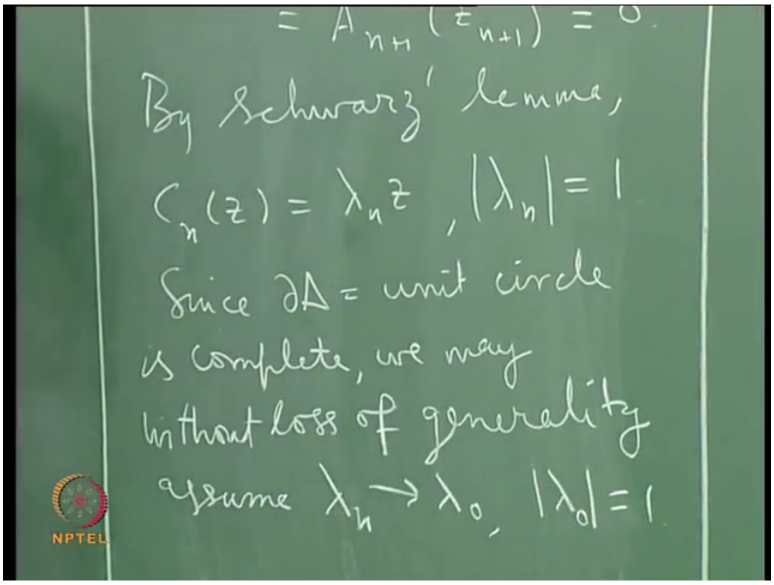

So, by Schwarz's by Schwarz's lemma you see C n of z has to be lambda n of z lambda n times z where lambda n is a is an element with modulus 1 namely, it is a rotation about the origin.

The only automorphism of the unit disk which preserve the origin or rotations that Schwarz's that is one version of Schwarz's lemma which we have already used and we are again using. Now you see this of course, the c ns are all from this c ns I can of course, get hold of an infinite subsequence. And therefore, I get in I get in infinite subsequence of lambda n's these are all points on the unit circle and you know that the unit circle is complete it is compact. So, I can extract from this a subsequence that converges 2 a point on the unit circle. And therefore, without loss of generality I can assume that the lambda ns tends to lambda naught with mod lambda naught equal to 1. So, I am using the completeness of the unit circle.

So, since dou the boundary of delta is equal to unit circle is complete we may without loss of generality assume that, lambda n tends to lambda naught and of course, with mod lambda naught is one. So now, what I want you to do, is I want you to do look at this sequence in between if I call this sequences h n. Then my claim is that that will that will give me a sequence which has an accumulation point, and what is the accumulation point the accumulation point is just gotten by putting letting n tend to infinity. So, let me write that down. And that will give me the contradiction that I wanted and that that will complete the proof.

So, let me write it down put h n to be the thing in the middle g n plus 1 inverse composition g n.

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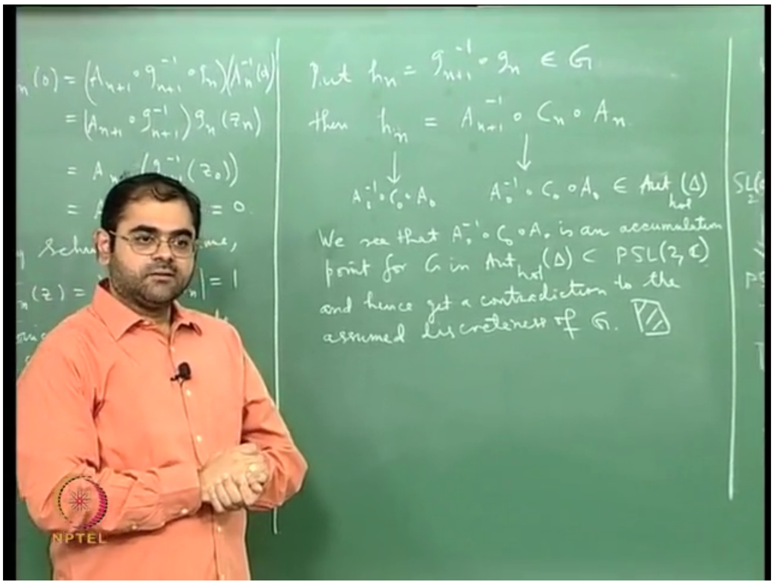

This is going to be a sequence in in G. Then you can see that you see h n well, h n is going to be it is just you know it is A n plus 1. So, I just compute what h n is using this definition. So, it turns out to be A n plus 1 inverse composition C n composition A n. And this tends to what this tends to as n tends to infinity, but as n tends to infinity you see a ns of course, a ns as n tends to infinity n tend to a naught and C n as n tend to infinity n tends to c naught. So, this tends to a naught inverse composition c naught composition a naught which is which is the point of which is the point of which is the of course, in automorphism of holomorphic automorphism of delta.

So, h n tends to this element. And of course, h n does contain and an infinite subsequence of distinct elements, because all the G ns are all distinct. Because you see all the G ns were chosen in such a way that g n z n is z naught, alright. And g m z m is also z naught. So, g n z n is g m z m. And you know z n and z m are distinct therefore, g n and g m have to be distinct. So, the G ns are all distinct alright. So, what you have found is you have found, certain subsequence of elements of G which has a limit point in the automorphism group of delta which is of course, this is again a subgroup of PSL 2 C. So, you have contradicted the assumption that G is discrete. So, that finishes the proof.

So, let me write that down. So, we can we see that a naught inverse c naught a naught is an accumul is an accumulation point for G in automorphisms, the a subgroup of holomorphic automorphism delta which is the subgroup of course, PSL 2 C. And hence get a contradiction to the disc assumed discreteness of G. So, that finishes it and as I have told you, I do not have to prove 2 implies 3 and 3 implies 4 they are trivial 4 implies one was already proved. So, the moral of the story is therefore, that. So, long as you are looking at a Fuchsian group, there is no different between the discreteness and kleinianess. And since PSL 2 z is indeed a Fuchsian group, and it is discrete it is Kleinian therefore, it will act properly discontinuously, and if you take the upper half plane, and you go modulo that group then the quotient will be Riemann surface. So, that finishes the proof that you can put a Riemann surface structure on the upper half plane modulo PSL 2 z.

Now, the rest of our discussion will try to show that this Riemann surfaces actually non-other than the complex plane, with the standard holomorphic structure. So, the original statement was you mod PSL 2 z is on the one hand bijective to the set of you know holomorphic isomorphism classes of complex store I that is what we proved. And now what we have proved is that the set of holomorphic isomorphism classes of complex store I itself is a Riemann surface as which comes as a ramified quotient of the upper half plane. And what we now are going to prove is that this Riemann surface is none other than the complex plane itself.

So, that is what we are going to do next. So, we will stop here.