An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-dimensional Tori and Elliptic Curves

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Lecture - 27
Quotients by Kleinian Subgroups give rise to Riemann Surfaces

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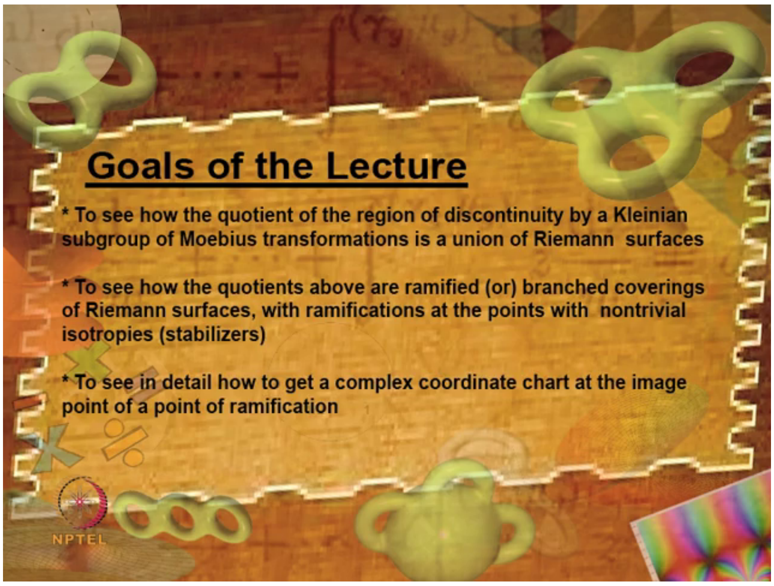

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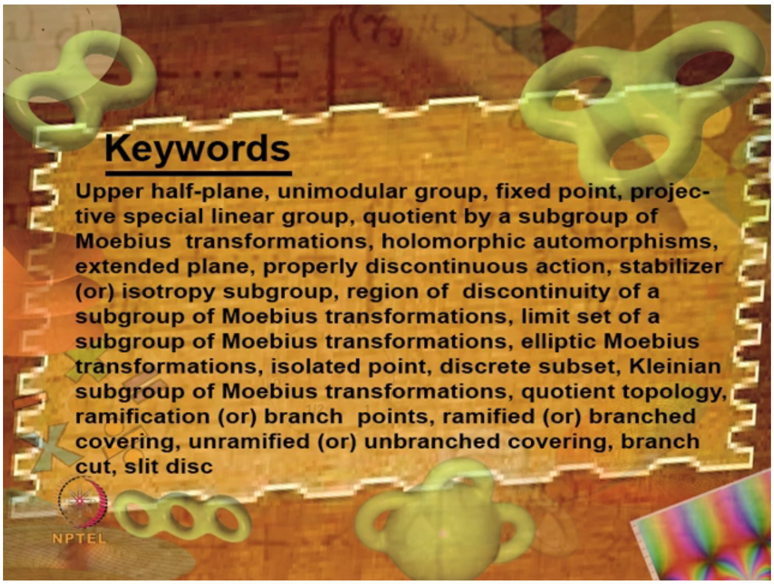

So, as you will recall what we are trying to do is trying to get hold of a Riemann surface structure on the orbits of onset of orbits for reaction of PSL 2 Z on the upper half plane and so the problem was that PSL 2 Z acts with fixed points. So, we defined properties of a group of Moebius transformations that will allow us to form a quotient by that group.

So, let me just recall very quickly what we did last time.

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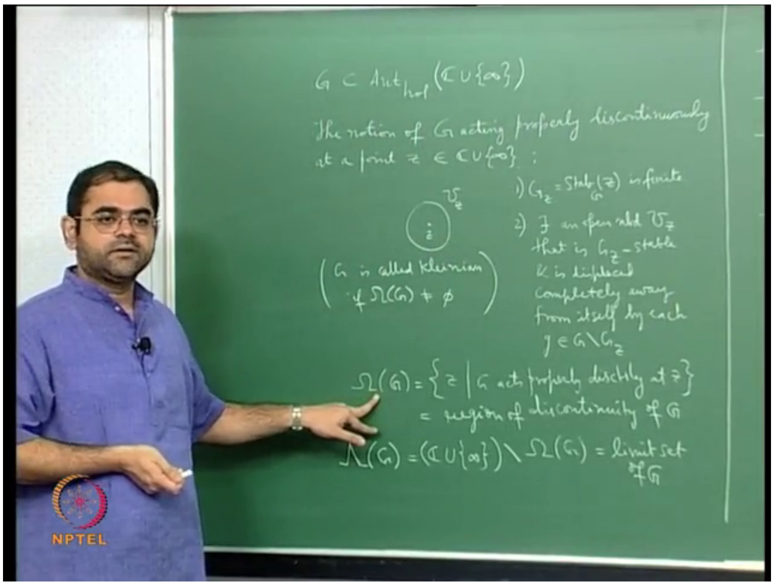

So, we have G is a subgroup of Moebius transformations automorphisms, holomorphic automorphisms of the extended complex plane and of course, PSL 2 z is one such. And for G we defined what is meant by the notion of G acting properly discontinuously at a point z in C union infinity.

So, this we define this notion. So, this notion was basically if you recall the definition as follows. So, well if z is a point here this is the point we are looking at in the extended complex plain then the condition was number one the set of points of G which leave z fixed namely the stabilizer subgroup of z has to be finite. So, G sub z stabilizer of z under G is finite; that means, they are only finitely many elements of G which leave the point fixed.

The second thing is there exists an open neighbourhood U z that is G z stable and is displaced completely away from itself by each element of G which is not in the stabilizer. So, these were the conditions that defined the notion of group of Moebius transformations acting properly discontinuously at a point there are only the stabilizers are finite and every point and there is it that is a open neighbourhood of the point which is mapped onto itself isomorphically by every element in the stabilizer and there are of course, finitely many doing this and for all the other elements this neighbourhood is completely moved away from itself. So, this was the condition, these were the conditions that defined when G is acting properly discontinuously at this point.

What we proved last time and then of course, you know we defined omega of G will be the set of all z where G acts property discontinuously at z. So, properly, I am using abbreviated notation d i s c t s l y for discontinuously or rather shorten notation. So, this is a set of points where the G acts properly discontinuously and this was called the region of discontinuity, discontinuity of G and its complement in the extended complex plane was called the limit set of G.

So, lambda of G is the complement in C union infinity of the region of discontinuity and this is called the limit set of G. And I must remind you that there is also a notion of G acting properly discontinuously when the action is free namely if G has no fixed points if G has no fixed points then of course, every stabilizer is trivial and in that case the condition reduces to finding a neighbourhood which is completely displaced by every non trivial element of G and that is by every element of G which is different from the identity.

So, for example, if you remember I told you that this definition in the case of a topological space or a Riemann surface for a group of holomorphic automorphisms of the surface or of the space of course, if you take a topological space you have to take homeomorphic automorphisms, continuous automorphisms then in that case I told you that going modulo of the group will produce a regular covering which it will produce a regular covering. Therefore, the group the space to the quotient is a covering space all, right.

The point is that we are we are trying to divide the upper half plane by the by PSL 2 z and PSL 2 z has elliptic elements and elliptic elements are elements which have fixed points in the upper half plane. So, we need to we need a definition of a group having a certain property which will still allow us to get an quotient which is a Riemann surface even though the group has fixed points and the required property is the property of the group acting properly discontinuously which is what I am which is what I have been talking about in the last a couple of lectures. So, what we proved in the last lecture was that you see the omega of G is G invariant is a G invariant open subset of C union infinity.

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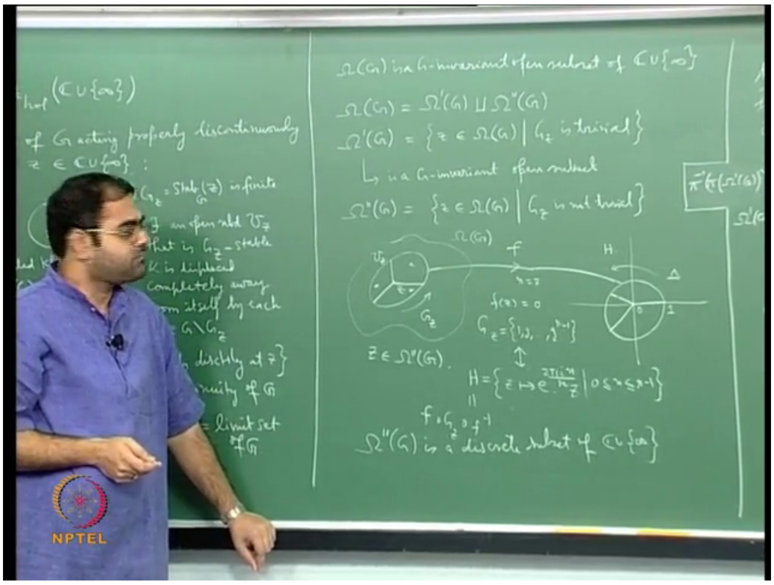

We proved that it is a G invariant open subset of C infinity and in fact, this is very easy to see because it is very easy to see when you are looking at points of omega of G which have trivial stabilizers. For points of omega which omega of G which have non trivial stabilizers if you remember we used Schwarz lemma and Riemann mapping theorem to get a picture of how the action of the stabilizer on that neighbourhood looks like and there was a lemma which said that the action looks like the finite the action of a finite group of rotations on the unit disk.

So, well and we also, so I should say that omega of G is broken up if you want into let me say omega prime of G and disjoint union omega double prime of G where omega prime of G is the set of all z in omega G such that G z is trivial this is a set of points of, these are the set of points where the stabilizer is trivial. In other words these are this is a complement of the fixed points the points which are fixed by some non trivial element of G.

So, and of course, so in fact, this if you take a point where the stabilizer is trivial then you are going to get an open neighbourhood surrounding that point which is displaced by every non trivial element of G and its of course, clear that every other point in that neighbourhood is also a point of this type it is also a point which has a trivial stabilizer and at which G acts properly discontinuously because the same neighbourhood will help to verify the condition of the action being properly discontinuous. So, this subset this is and of course, it is also an open subset.

So, this is a G invariant open subset of course, when I say G invariant I mean a subset that is mapped by G onto itself isomorphically of course, this is G variant open subset and well and what about omega double prime G omega double prime G is well the set of all points where G z is not trivial these are all the points where there is a non trivial element of G fixing that point. So, stabilizer is non-trivial, but of course, because of the properly discontinuous action that stabilizer is anyway finite and well if you take.

So, let me draw a small diagram the picture is like this well if this is well omega of G and if you have a point z in omega of c then what we saw last time so in fact, if you want if z is you take a point z in omega double prime of G. What we saw was that we can find a neighbourhood of this point D, D if you want D sub z. In fact, let me call it as U sub z I can find neighbourhood of this point open neighbourhood of this point and a map a holomorphic isomorphism of that neighbourhood with the unit disk.

So, this is a unit disk with the point being mapped to 0 f of f of z is 0. And if you see the group G sub z maybe the finitely many elements of G which move I mean which fix z that subgroup the stabilizer subgroup the way it maps this neighbourhood onto itself via this map can be compared to the action of a group of rotations of the unit disk about the origin. In other words if I write if the cardinality if G z is it is has cardinality r may be it consists of r elements it will be a cyclic group and if a generator is let us call the generator as G then you this group will be up to G to the r minus 1 this will be the elements of the group where I write one for the identity element of the group please do not confuse into the complex number one these are all automorphisms these are all Moebius transformations.

So, this is your group this is your stabilizer subgroup it has n r elements and is cyclic and that is cyclic also came from our discussion in the previous lecture and you see this G is via this map which is a holomorphic isomorphism. This G is can be identified with the group H of rotations this group H of rotations is just going to be you know all those. So, rotations about the origin and so that you know it is a rotation group of time of with r elements therefore, it has to be given by H z going to e power 2 pi i n by r where well 0 less than or equal to n less than or equal to r minus 1 this is your rotation group and this is, this is z going to this time z multiplication by e power 2 pi n by r is well rotate by the angle 2 pi rotate by the angle 2 pi n by r right n varying from 0 to r minus 1.

And what is H under this map this H is actually you know it is just a conjugate you apply f inverse because f is a holomorphic isomorphism then you take G z and then again apply. So, it is a conjugate via this map. So, the point is that H is acting here and the action of H the group of rotations is if you look at it in terms of this map then it translates to the action of G of z here.

So, in particular if for example, if you want me to draw the picture for r equal to 3, if you want then you know the rotations are going to give me well let me draw it like this. So, well if I take a sector like this then I have this at 120 degrees and another at 240 degrees then there are three sectors and, r is 3. So, the action here is this sector moves this sector this sector moves this sector right. And that that will translate here too well something like this, this is how it will move. So, G the action of G z on U z can be is apply some of them just the action of H on delta.

So, this is the picture that you get of the action of the stabilizer at a point with non trivial stabilizer and this picture tells you 2 things it tells you that the those points which have non trivial stabilizers they are isolated because you given any such point which is with non trivial stabilizer if there is a whole neighbourhood where every other point has trivial stabilizer because you take any point here which is different from z then well it is of course, going to be bound by every element of G different from G z because the whole neighbourhood is going to be displaced away. So, every element of G different from G z is not even no is never any, is never is no way going to fix this.

So, you have to only look at elements of G z, but elements of G z are going to act like rotate like rotation. So, this point will go to some point here under and then it will go to some other point here. So, these are all going to be distinct points. So, there is going to be no stabilizer for any point different from z in this neighbourhood what it tells you is that every such point here has a neighbourhood which actually lies here and therefore, the this set of points with non trivial stabilizers is a discrete subset.

So, this set is this is a discrete subset of the extended complex plane right. So, omega double prime G is a discrete subset of C infinity. So, this is a picture that we have of how a point which is having a non stable a trivial stabilizer how these non trivial stable stabilizer acts in the neighbourhood of the point fine. Now, what was the aim? The aim was to well the aim was to divide by such a group. So, what I want to show is that you take this region of discontinuity and you divide it by the group what you get is a Riemann surface.

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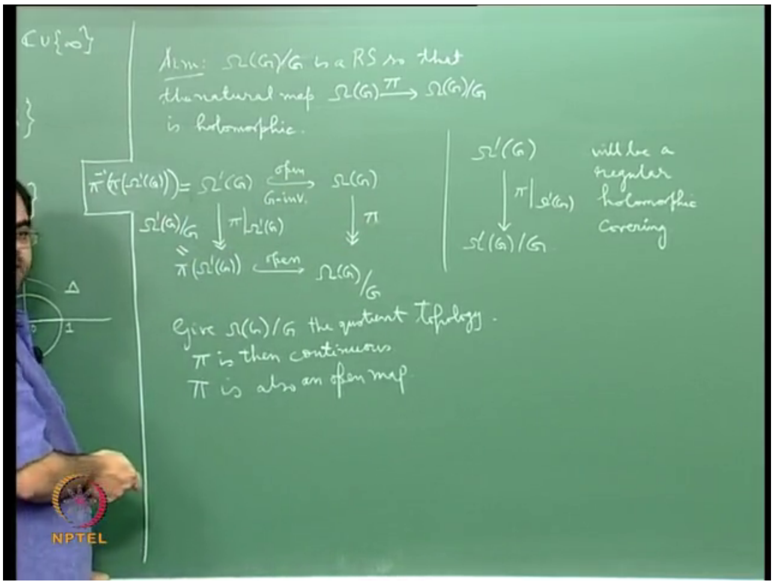

So, the aim now is omega G to omega. So, omega G mod G is a Riemann surface so that the natural map the natural quotient map omega of G to let me call that map as pi to omega G mod G is holomorphic. So, I am going to tell you why this definition is a good definition because it is a good definition because you can divide omega G by G to get really a Riemann surface.

So, if I do this then the only thing I will have to show is that PSL 2 z this acts properly discontinuously on the upper half plane and then you will be convinced that U mod PSL 2 z certainly a Riemann surface and the natural map from U the upper half plane to U mod PSL 2 z is holomorphic. Of course, in all these things I am assuming that G is Kleinian. So, I should remind you, so G is called Kleinian if omega of G is non empty and of course, we are working only with Kleinian groups, but I do not want this be the empty set right.

So, well, so let us let us look at this map. So, what we will do is look at omega of G to omega of G mod G. Of course, you know when I write omega of G mod G what this means this is the set of orbits this is a set of orbits of G in omega G this is the quotient map all right and of course, it is a surjective map I am calling this as pi, it is a surjective map. And the first thing I want to do is I want to make this at least a topological space and you know the natural thing that one would do is to put the quotient topology, you put the quotient topology so that this map automatically becomes continuous what is the quotient topology is the quotient topology is the is a topology that is given by the condition that any subset here is open if and only if its inverse image above is open.

So, give omega of G mod G the quotient topology topology give it the quotient topology and then it is very easy to see of course, because if given the quotient topology this map becomes continuous pi is then continuous of course, and what I claim is again pi is actually an open map, pi is an open map. Namely, it takes open sets to open sets and that is something that is very easy to prove because you take any first of all take any set here you take its image below and then if you take the inverse image what you are going to get is just translates of the set by elements of G.

In particular if you started with an open set here you take its image below and then you take the inverse image you will get all translates the union of all translates of that open set by G, but that is again an union of open sets because a translate of an open set by an element of G is another open set because G is every element of G is acting by holomorphic automorphisms therefore, it is an open map.

So, in particular you see what will happen is this will tell you that if I take the subset omega prime G which is an open G invariant subset. Then its image pi of omega prime G will be the same as well it will be the same as this will be open of course, this will be an open subset and pi of omega prime G is what is this is just and this is nothing other than omega pry or pi of omega prime G is just omega prime G modulo G. That is because you see if I take the inverse image of this the in inverse image of this will be again omega prime G the inverse image cannot contain points in omega double prime G because of the description of omega prime G and omega double prime G.

So in fact, this omega prime G I should write is pi inverse. So, well I do not have space, maybe I make a little bit of space here the duster is little more dirtier. So, let me make a little bit space and write omega prime G is pi inverse of well pi of omega prime G because of the fact that every point in omega prime G is a point with trivial stabilizer and you cannot get a point outside of omega prime G because they serve these are all going to be points with non trivial stabilizer and appointed trivial stabilizer cannot be in the orbit of a point with a non trivial stabilizer and conversely because all points in an orbit if you take the stabilizer they are all going to be conjugates.

So, if one of them is trivial every one of them is trivial all right. So, well because of this reason this is actually going to be omega prime it is just going to be omega prime G divided by G it is just taking orbits in omega prime G, G orbits in omega prime G. Now I want you to remember that the G action on omega prime G is free because after all omega prime G is a set of points where G is its all those points where stabilizes a trivial; that means, G acts freely and we always already seen in this case that these are going to be a regular covering.

So, what you must understand is that this piece omega prime G to for this open subset this quotient map. So, this is just, this is just pi restricted to omega prime G. So, this quotient map pi restricted to omega prime G from omega prime G to well it is quotient this is going to be a regular covering this is going to be a regular holomorphic this is going will be a regular holomorphic covering.

So, the point is that when you take this quotient there is really no problem in getting a Riemann surface on this open sense the problem is only at those points it is only at those points where the stabilizers are non-trivial. These points are they are special they are called ramification points because these are the points where the where your quotient map will no longer be a covering in fact, this quotient map is going to be a covering exactly outside the d exactly on this open side which is this minus a discrete subset of points and it is called a ramified cover and a covering the usual regular coverings I mean the usual holomorphic coverings are in that sense under ramified because you know of course, you know that no deck transformation can have a fixed point.

So, it stabilizes are all trivial. So, the problem is only trying to the problem is only trying to you know give, what is the problem. So, a problem is you take a point like this a point like this is going to be in omega double prime C, for a point like this you take the if you take its image in the quotient below the problem is to give a holomorphic coordinate at that point that is the whole problem and that can be done very easily as follows. So, well to understand that clearly let us try to you know let us try to look at let us try to look at this situation.

So, let us take let us take the situation where we are trying to divide the unit disk by a group of rotations because after all around a point which is having a non trivial step stabilizer the picture looks like this. And you know trying to go trying to go a modulo a neighbourhood of this point going model G the same as going model G z you have to essentially divide only by G z. So, what you should actually study is you the quotient U is a mod G z discussion by a finite group and U z mod G z is literally the same study delta mod H because of this holomorphic isomorphism f.

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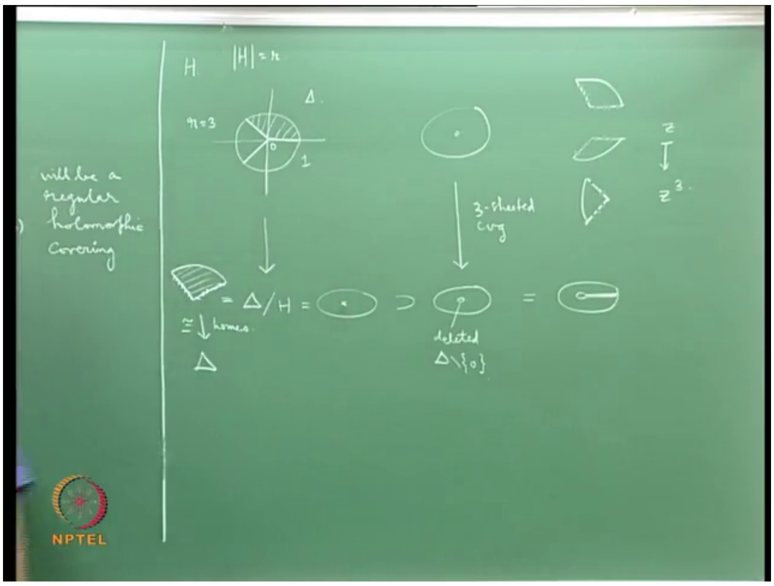

So, let us look at this delta mod H. So, the situation is like this. So, here is my delta. So, let me write let me draw a diagram. So, I have unit disk delta and I have H and well you all know that if I take delta mod H; what am I going to get. So, you see well let us let me draw the picture for let us say H having three elements. So, H then contains multiplication by 1 omega and omega square which are the omega is a complex cube root of unity H is the three dimensional H is a three group with 3 m s right.

So, you see then you know that if I take the quotient of delta by H for example, r equal to three cardinality of H is are then you know what you are going to get this is going to be see I all I have to do is I will take any one of these regions I will take any one of these regions then these there are three sectors and I have to just take any one of them all righ. I have to take a sector like this and well I have to identify these two radii these two radii have to be identified and if you check that this is homeomorphic to delta itself because after all you know I can simply stretch and stick it together I will get back delta.

So, what you must understand it you take delta and you divide by H a finite sub group finite group of rotations what you are going to get is topologically again delta you are again going to get the unit disk. So, the question is in this case which is a finite group it is very it is very clear to see that the action of H of course, elements of H or Moebius transformations multiplication by some constant complex numbers there Moebius transformations and it is very clear that H is going to act properly discuss easily because you can see that this is the only point which has stabilizer non trivial stabilizer and every other point as trivial sub stabilizer.

So, H is certainly acting properly discontinuously alright. So, the picture is like this. So, the question is if you take delta mod H it looks like we again looks like delta and then the question is on delta mod H, how are you going to give me a how is one going to get a coordinate. So, for that to analyze that what one does is one looks at it little carefully. So, what you do is well this is let us look at this as delta at least topologically for the moment that is if you put the quotient topology you will get delta there is no doubt about it, but you have to make it into Riemann surface. So well, at that is this point which is the image of zero it is this point that is a that is a troublesome point that is a point at which it is its immediately not clear to me as how to get a complex coordinate neighbourhood because why is that the problem because if I throw away that point.

If I see, if I take if I take out that point, I put a small circle here saying that this is deleted I have deleted that point, it is a punctured, this is delta well this is delta minus 0 if you want it thrown out I am calling this point also below a 0 I am identifying this with delta I am calling this point also a 0 all right and I throw out that then you know this is a then from here to here this is a 3 sheeted covering this is an r sheeted covering. So, in this case r is 3. So, it is a 3 sheeted covering and well it is a 3 sheeted regular covering that is because that is because of this fact here from omega prime G to omega prime G mod G that is always a regular covering that is because G acts in this case G is replaced by H and this is the domain am looking at which on which it is acting.

So, this part is a nice 3 sheeted regular covering and what are the sheets if you want to get this sheets. So, this is where the point is, the point is well this point above this point above is called the ramification point it is called the ramification point. And how do I get those 3 sheets. Well, it is very simple what you have to do is you have to well if you want to get the 3 sheets you cannot get them globally you cannot get the 3 sheets globally unless you make a branch cut. So, what you do is you do not only delete this point, but you also make a branch cut here. You take the slit open this unit disk and the slit you cut out also the origin which is a troublesome point then you can see that the three pieces of the cover are exactly these three open sectors, they are the three sheets you can easily see that you know I can get you know I can get these three as the three sheets of the cover and this well you can at least guess this and well and, you know these well these things are not included I mean the radii are not included and of course, I am also drawing the circumference circle that is also not included because I am worried I am taking only the open disk.

So, you see these are the three sheets of the curve and what is it, what is the covering map? It is simply is z going to z cube it is simply the map e z going to e z cube that maps each of these three sectors homeomorphically, holomorphically onto this slit open disk. So, you see that gives you that gives you the clue. The clue that you get from this is what you do is you make this branch cut and once you make this branch cut then this can be identified with one of these pieces once you make the branch cut then you can identify with one of the open sectors and then you follow it by the map e z going to z cube the resultant thing will give you a homeomorphism of this slit open unit discount itself.

So, and that homomorphism you can complete it by including the seed slit to get a coordinate map to get a coordinate and that is how you get a coordinate at that point. So, this is exactly what we will be doing in the quotient case. So, well let me draw a diagram and explain. If you understand this very well in the general case if H is r then I will have to take z going to z power r alright. So, the whole point of the discussion is that unit disk modulo finite group of rotations is the simplest model for a ramified cover with the ramification at the origin which is ramified point. Outside that point what you get is a nice covering space at that point is ramified, but still even though that point is a point of ramification there is no problem in giving complex coordinate neighbourhood at the ids image. So, this is the simplest situation of a ramified map.

And what this Schwarz lemma thing says is that this how it is going to always look whenever you have a finite acting on a point on a Riemann surface. So, let me draw another diagram here. So, that it is easier to visualize things. So, you see, so I am in the following situation.

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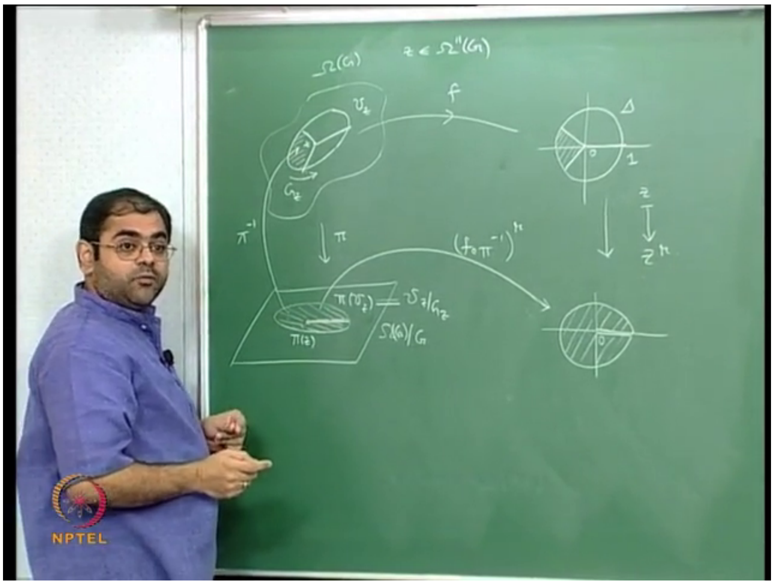

Here is my well omega of G and there is a point z there is a point z which is a point of omega double prime of G. So, it is a point with non trivial stabilizers. So, I am just redrawing the diagram here. So, I have a neighbourhood. So, this is z this is U sub z and I have something like this and well that is this, that is this holomorphic isomorphism f of this neighbourhood with the unit disk delta with of course, z going to 0 all right. And I have the quotient map pi which is good and I am essentially trying to look at this space here which is omega of G on G quotient I am looking at this point pi of z and of course, you already know that pi is an open map. So, pi of user is going to be well an open set going to be an open neighbourhood of pi of z all right.

Now, the point is I want to coordinate here. So, how do I get the coordinate here what I do is well the, what is acting here is my G sub z and you must understand that this pi of U sub z is actually use U sub z mod G sub z. So, this is see the image of pie under U z is essentially taking U z and going model only G z it does not make sense to take use and go modulo G because G will move it, what acts on this open neighbourhood use it is only G z. And pi of U z is going to be there for this is going to be U z mod G z and what am I going to do I am going to do the following thing I have this map e z going to e z power r and this map is going to map unit disk onto itself and this is a map that is ramified at the origin.

Now, what you do is the following you I am drawing diagrams for r equal to 3. So, you know you make you make a slit here along this line then well I will get three sectors here and these three sectors are going to be mapped to 3 sector like regions all right and well G the this G z is going to identify all of them it is going to just identify all these the interior points of these sectors they are all going to be identified and the and the boundaries are going to be identified. So, the result is again something that is dislike and what am I going to do well I also get a branch cut here.

Now, you see if I remove this branch cut I mean once I make this branch cut then you see pi inverse I can define pi in verse on to say one of the sectors one of the open sectors because it is an open sector because I have cut out this and I have cut out this and that is because I have cut out this. And now if I follow it by f I am going to get one of the sectors here one of the open sectors here if I follow it by this map is it going to z power r I am going to get the whole slit open unit disk. So, the moral of the story is if I take this composite map which is apply pi inverse then apply f and then raise it to the power of r then that maps this slit neighbourhood hope and it is going to be holomorphic mind you pi inverse the moment I make a slit here the moment I make a slit here it is a regular it is a regular cover.

So, there are three sheets in fact, there will be r sheets if the cardinality of G z is r. So, I will have a I will have local in verses which are holomorphic maps because it is a regular cover. So, this is holomorphic, this is holomorphic, this is holomorphic, the result is I get here holo. So, I will get what, I will get is it where I should not say this holomorphic it is if I remember that this slit region is in the Riemann surface omega prime G mod G.

So, this is holomorphic, this holomorphic, this is holomorphic. So, this is holomorphic and this map can be extended to homeomorphism by including this slit. So, I get a homeomorphism of this with the unit disk that gives me a coordinate at this point. And it is its very obvious that this coordinate chart at this point is going to be compatible with any other coordinate chart because I have to test only at other points if I take a point different from this you have to do a little bit of work and you can easily see that this coordinate chart is going to make this neighbourhood into a Riemann surface and that Riemann of a structure when you from that Riemann surface if you get the open subset which is got by deleting this slit the resulting wave man surface structure is exactly the remain structure surface structure on this restricted to that open set.

So, it is not going to create any issues of compatibility. So, the moral of the story is that at this point which is image of a ramification point I do get complex coordinate. So, in this way every point of omega G by G has a local complex coordinate and these charts are all compatible, so this becomes a Riemann surface and this becomes a covering which is called a ramified covering. And the covering is ramified exactly at the points above where you have non trivial finite stabilizers and outside those points you get an open subset where the covering is a un ramified covering in the best sense. And how does this ramified cover look at a point where which has finite where the stabilizer is finite it looks exactly like the taking the unit disk and going modelo a finite number by finite number of rotations that form a rotation group. So, that is the, that is upshot of all this.

So, at least this discussion tells you that indeed G being properly discontinuous allows you to make omega G mod G into Riemann surface. Of course, in this discussion I have assumed that omega of G is connected because you want us you know p would like spaces to be connected, but in general omega G quill break up into components and you can do this you can carry out this argument for each component and then in general what you will get is omega G mod G will be it will be a union of Riemann surfaces such that the quotient maps from each component to its quotient are going to be you know. So, I will have to if this is not connected I have to take connected components, I will take a connected component and work there all right. So, this tells you that G being a group with a properly discontinuous action group of Moebius transformations does allow you to cook up a Riemann structure on the region of discontinuity modulo that group G.

So, now, what I need to do is only one thing I need to show somehow that the action of PSL 2 z on the upper half plane is properly discontinues. That is in other words I have to show that for the action if I put G is equal to PSL 2 z I have to show that omega G contains upper half plane that is what I have to show once I do that then you can be convinced that p you upper half plane model of PSL 2 z is indeed Riemann surface. So, we will do that in the forth coming lectures.