An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-dimensional Tori and Elliptic Curves

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Lecture –26
Local Actions at the Region of Discontinuity of a Kleinian Subgroup of Moebius Transformations

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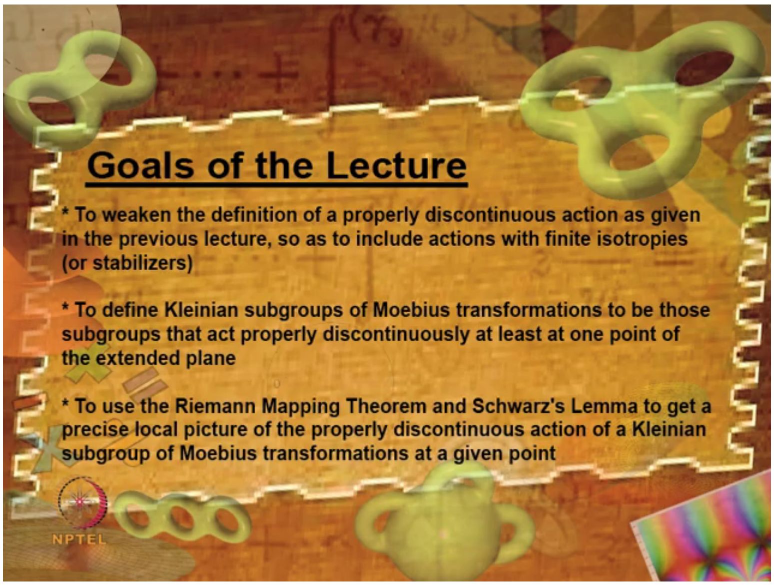

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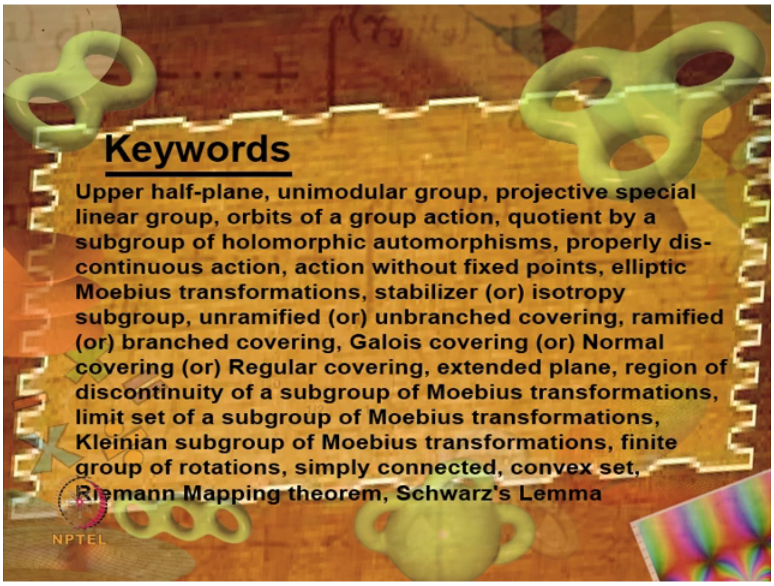

Let us recall what we are trying to do is try to give the structure of Riemann surface to the upper half plane, modulo, the action of the Unimodular group namely PSL2Z. So, you we take the projective special area group these are Moebius transformations with integer entries they preserve the upper half plane and we want to look at the orbits for these action set of orbits and want to say that that is Riemann surface, and then we want to show finally, that this Riemann surface is biholomorphic to the complex plane.

The first question that arose was how do you make this into a Riemann surface how do you give a Riemann surface structure on U mod PSL2Z, more generally the question was suppose you had say Riemann surface and you had a group of holomorphic automorphisms of the Riemann surface, subgroup of holomorphic automorphisms; can you divide by that subgroup to get a quotient which is also a Riemann surface, that was the original question.

We have seen examples of this for example, in the case of the universal cover what happens is that the you have the universal covering space of a Riemann surface and then if you go modulo the deck transformation group which is identifiable with the fundamental group of the base space, then the quotient is exactly the base space and then more generally I told you that this is happening because of certain property of the action of the group which is called properly discontinuous action. But then, this definition of properly discontinuous action that I explained in the last lecture was a definition that involved deck transformations and you know that this definition presupposed that the group was acting without fixing points.

It is a, what it says is that if you have a Riemann surface or more generally even a topological space and if you have a group of automorphisms which add properly discontinuously, then you can divide by that group namely you can take the set of orbits under that group and that will automatically become a Riemann surface if the original space was already a Riemann surface and this definition of properly discontinuous action was a definition that presuppose that there were no fixed points because this definition of properly described his action was that given any point there is a small neighborhood of the point, which is completely moved by every element of g different from the identity is completely moved away from this from itself by any element of G different from, any element of the group which is different from the identity, but as, but then this is not helpful for us directly because when you are trying to look at the action of PSL2Z on the upper half plane, there are fixed points, there are elliptic Moebius transformations for which there are fixed points. The group PSL2Z is not acting on you with fixed points and therefore, you cannot do you cannot simply use this theorem 2 divided by PSL2Z. So, what we need is we need a slightly more relaxed definition of what a properly discontinuous action means, a definition which will also help you to get quotients when there are fixed points which is what we need.

Let me recall that definition. The definition was as follows, we had, so X maybe I will confine myself to.

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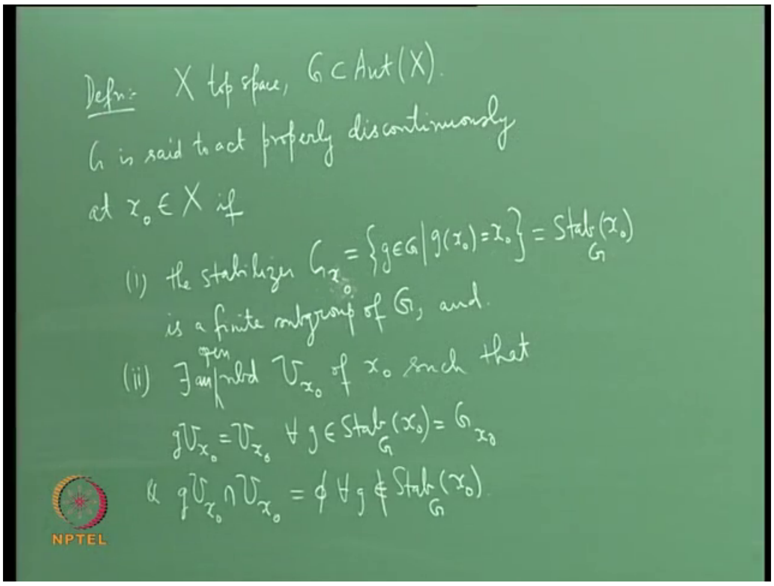

Let me make the definition, let me recall this, you have X topological space G sub group of automorphisms of X of course, instead of topological space I could have of course, taken also Riemann surface and in this case in that case G would be as a group of holomorphic automorphisms of X. We say G is said to act properly discontinuously at x naught if number 1, this is the first condition is about fixed points. So, you know in the earlier definition of properly discontinuous action, there were no fixed points every non trivial group element every non trivial automorphism was not supposed to fix any point and this is how this is exactly how your every non trivial deck transformation acts deck transformation, no deck transformation different from the identity can have a fixed point.

Here, that means the set the set of group elements that fixed point has to be trivial, this was what was there in the earlier definition, but now what we will do is we will relax that and say that the set of elements of G which fix this point need not be trivial, but it has to be finite. That is the first condition the stabilizer G x naught which is the set of all g belonging to G such that g of x naught equal to x naught is a finite sub group of G, of course, the stabilizer will be a subgroup by definition and all we want is that it is finite and the case when there are no fixed points that is when the action a fixed point free this stabilizer has to be trivial. That is it contains is just that is a subgroup which contains only the identity element the identity transformation then and the second condition is about this, the existence of a neighborhood for which the proper discontinuity of the action is demonstrated.

In the earlier case when the group is acting freely the properly discontinuous definition was you have a neighborhood which if you operate by an element of G it is completely displaced from the original neighborhood that is the image neighborhood does not intersect the original neighborhood. Now what we will do is we will relax that we will demand that only of elements outside the stabilizer and we will also demand that neighborhood is preserved by the stabilizer of course, we cannot demand that of elements in the stabilizer.

Because after all if there is going to be G which is going to fix x naught then the any neighborhood of x naught will also contain x naught when it is operated upon by G. So, here is the second condition which is also a relaxation of the earlier definition that is there exists a neighborhood U sub x naught of x naught of course, I should say open neighborhood, that is an open neighborhood of course, whenever I say neighborhood of course, I mean open neighborhood such that such that g for every element in the small g in the in the stabilizer subgroup. This is the in fact; let me also write this as tab stabilizer of x naught in G which is just G sub x naught.

For every element in a stabilizer this neighborhood is preserved and for every element outside the stabilizer the effect the image of this neighborhood under that element is does not intersect the original neighborhood. Let me write that out g Ux naught intersection U x naught is empty for all G naught in stabilizer, you see this is a more weakened definition of what is meant by properly discontinuous action and finally, what we are trying to do or what we are going to prove is it the action of PSL2Z on the upper half plane satisfies these conditions.

That is something we are going to see and we are also going to see that whenever you have a group of Moebius transformations which acts on a domain in the in the external complex plane which is in this sense properly discontinuously then you can divide by that group of Moebius transformations and obtain a Riemann surface. The only difference between this situation the early situation was that in the earlier situation one gets a real covering of Riemann surfaces. In fact, we get a regular covering as I was explaining to you in the last lecture, but in this case what you will get, is you will get something that is a covering only on an open set below, but it will have, but there will be a boundary where the map is ramified.

There will be a ramification locus and it will be what is called a magnified covering of frame of surface. So, what I want you to understand at the outset is the earlier definition of properly discontinuous action what happens is that when you divide by such a group with such an action what you get is actually a covering of Riemann surfaces whereas, if you use this definition what you will not get; what you will get is not a covering of Riemann surfaces, but what you will get is what is called a branched covering or ramified covering of Riemann surfaces. So, that is the difference and this branching comes because of the existence of isotropies stabilizers finite stabilizers, now I am going to just restrict my situation to you know Moebius subgroups of Moebius transformations acting on C union infinity the external complex plane, let me make this definition, I will make this.

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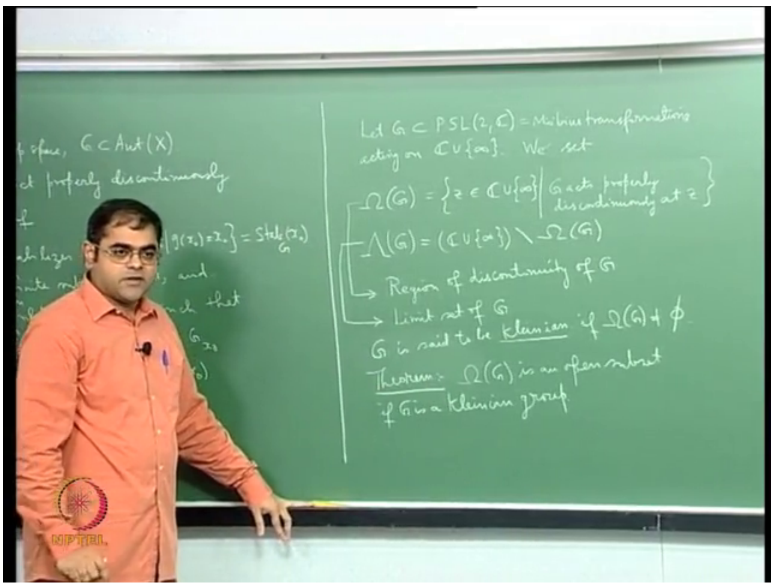

Let me write this let G be a subgroup PSL2C which are just Moebius transformations, let G be a subgroup of Moebius transformations acting on C union infinity.

This is the external complex plane if you want to think of it like that you can also think of it as a Riemann sphere which is P 1 where the stereographic projection now what we do is, we set omega of G to be the set of all z in C union infinity set of all complex numbers in the extended plane where G acts properly discontinuously at z. So, you see the first thing that we will have to establish is that this definition of properly discontinuous action has got some good properties so that is what I am trying to do. So, what I am doing is given a group G of Moebius transformations which will act on the external plane I am collecting all those elements in the external complex plane or at which G acts properly discontinuously. You see when we define properly discontinuously properly discontinuous action it was defined for the action of the group at a point.

There may be points where the group is acting properly discontinuously and there are going to be points where the group is not going to act properly discontinuously and of course, what we are trying to say is that the points where the group is acting properly discontinuously is the right set to work and one of the good properties that we are going to prove now is that that set is open, and of course, we are going to let me put lambda of G to be the compliment, this is the set of points where the group does not act properly discontinuously. It is just C union infinity from that I take away the set of points where the group acts properly discontinuously, there are names for this obvious names the set of points where G acts properly discontinuously is called the region of discontinuity of G and of course, the set of points, though it says region of discontinuity of G somehow the word discontinuous puts one off and one feels that something is bad, but actually here is something good.

The region of discontinuity is a very good region it is exactly the region where you can go modular G and get a Riemann surface. So, do not be misled by this name or by this terminology and capital lambda of G is called the limit set of G, the well the first, the here is a first let me say theorem. Before I say this I should say the following thing we say that the group is Kleinian group after Felix Klein who German mathematician who pioneered study of geometry interact I mean and it is interactions with analysis and algebra. We say the group is clean in if there is at least 0.1 where the group acts properly discontinuously all right, that is a definition, maybe I will write the definition first let me write that down G is said to be Kleinian if omega of G is non empty, there is at least 0.1 where the group acts properly discontinuously.

Here is a theorem, which tells you that the when a group acts the set of points where the group acts properly discontinuously for a Kleinian group namely the region of discontinuity that is actually an open subset. So, here is a theorem omega G is an open subset if G is a Kleinian group, here is the theorem, the point is that this fits in with the general philosophy that any good property which is defined at points should be an open property.

If you try to define you know holomorphicity for example, you if a function is holomorphic at a point then it is holomorphic in every point in a neighborhood of that point. These are all properties which are which you define a point they are good because they are kind of true for if they actually at a point then they are true in a small neighborhood of the point. That is exactly what we are saying here what we are saying is if you take a point of omega G then there is a whole open there is a small disk surrounding that point which is full of points again in omega G.

If so; that means, that the moment G is a Kleinian group that is the moment that there is you know that there is at least one point where the group acts properly discontinuously you know that there is a whole disk surrounding that point where the group is going to act properly discontinuously, so what is the proof.

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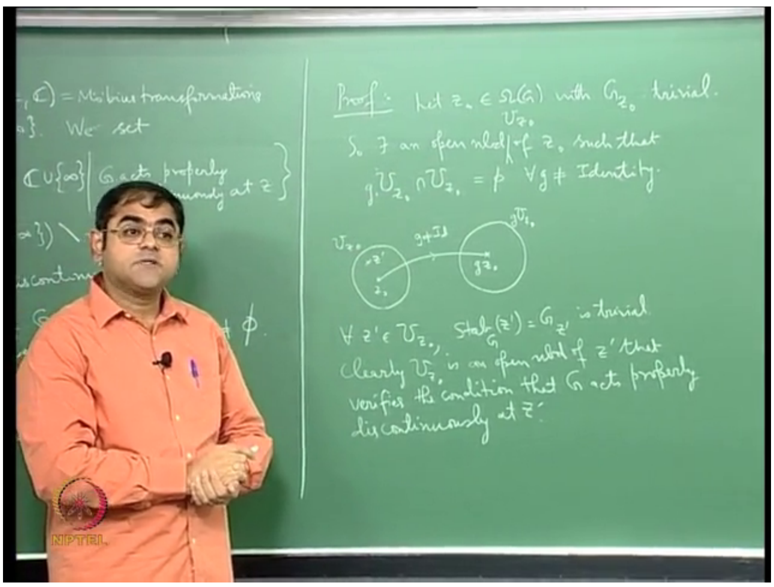

The here when we get into proof we will use some inputs from complex analysis, so here is a proof, first let us, we can consider 2 types of points in omega G recall a point in omega G is a point at which G acts properly discontinuously and the stabilizer is finite. Let us first dispose of the case when the stabilizer is trivial; let is it not be a point of omega of G with stabilizer subgroup trivial.

Now, it is very clear if you think about it that there is a whole disk surrounding z naught which all at which also the group will act properly discontinuously and all the points will have strictly stabilizers why is this, that is because you see if is it not is if the stabilizer is trivial. So, you look at the second condition here that is there is an open neighborhood use of z naught of z naught such that for every G mind you the stabilizer is trivial; that means, for every G which is different from the identity G times that neighborhood will not intersect that neighborhood.

Let me write that down, there exists an open neighborhood of z naught such that I should say use of such that g dot u sub z naught intersection u sub z naught or, but if you want I can write g dot U sub z naught or g of U sub z naught this means the image of U sub z naught under g, this is empty for all g naught equal to identity for all g different from identity there is a neighborhood like this. Now you see if I draw a diagram here is my z naught there is this there is a neighborhood of course, it need not look like a disk, but I am I am just drawing it for ease of representation. So, you see, here is my U sub z naught and if I take a g naught equal to identity and apply to this what I will get is, I will get a holomorphic isomorphic neighborhood which is namely U sub z naught and this will be a neighborhood of g of z and these 2 do not intersect.

It is very clear that you see if you take any z prime in this neighborhood then it cannot have it is stabilizer will be trivial for all for every z prime in U z naught the stabilizer subgroup namely which otherwise we write as g sub z prime is trivial. This is obvious because you know if there is a g element which fixes z prime then g of z prime equal to z prime, but you see g of z prime is supposed to be here all right, what it will tell you is that that cannot happen if g is not the identity elements, what it will tell you that every z prime in this neighborhood has trivial stabilizer and it is also clear that every non trivial element of g is going to push this neighborhood away from itself.

So, you see the same neighborhood ignored z naught will also serve as a neighborhood for z prime for verifying the condition of a properly discontinuous action of g at z prime. So, clearly U z naught is a neighborhood an open neighborhood of z prime that verifies the condition that G acts properly discontinuously at z prime. What is the moral of the story, the moral of the story is you take a point it trivial stabilizer you take a point of the a point in the region of discontinuity a point z naught where the group acts properly discontinuously and where the stabilizer is trivial then you can find a whole neighborhood where again the group acts properly discontinuously and the stabilizers are also going to be trivial. The moral of the story is if you take the subset of omega of G consisting of points with trivial stabilizer that becomes an open subset of the complex of the external complex plane.

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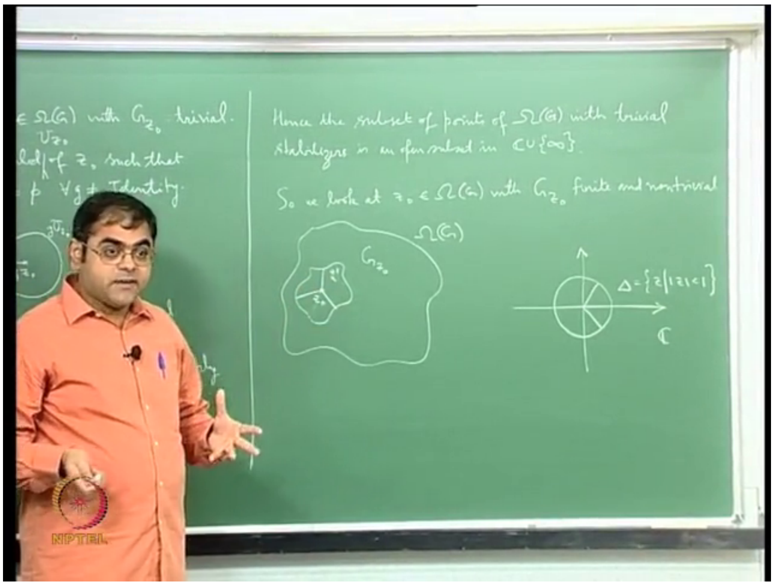

Let me state that hence the subset of points of omega of G with trivial stabilizers is an open subset in external (Refer Time: 25:13). We have dispensed off the case with the case of points with trivial stabilizer the more serious case non trivial case is that of points with non trivial stabilizer. We have to look at a point z naught at which the action of g is properly discontinues, but the stabilizer is not trivial now go back to this condition the stabilizer has to be finite, what happens in that case right. We need to, we look at z naught in omega G with G z naught finite and non trivial so; that means, you look at those points where the stabilizer is not just the trivial element, what happens in this case.

At this point some analysis has to be brought into the picture right, I roughly tell you what we are going to do what I am going to prove, you see, here is you know well let me suppose this is omega G all right and here is my z naught what I am going to tell you is, I am going to tell you that actually you see this that for a point z naught with stabilizer finite I am going to say that there is going to be a neighborhood of this z naught which you know it looks like, there is a neighborhood surrounding z naught which looks like the unit disk delta.

This is the unit disk in the complex plane, delta is set of all z such that mod z is less than one, what I am going to show is that you see there is a neighborhood of z naught which is contained in omega of G all right or let us not even say that for the moment there is a neighborhoods of z naught which looks like it is and the action of G sub z naught on this neighborhood, it is going to be a neighborhood which is going to be fixed by G sub z naught this finite group all right and it is going to be moved completely away by other elements of by elements which do not stabilize z naught elements of the group which do not fix z naught already such a neighborhood is available.

This U z naught is available already, what I am trying to say is that you can in fact, choose a neighborhood which actually looks like the unit disk and such that the action of G z naught on this neighborhood looks like the action of a finite group on this unit disk a finite group of rotations about the origin. So, what I am going to prove is that you can find this neighborhood here and in fact, a holomorphic isomorphism of this neighborhood with the unit disk, when I do that this G z naught which is a finite group it is going which is going to act on this neighborhood if I transport it is action via this identification of this neighborhood with the unit disk. Then the action of the group there will look like a finite group of rotations, you know for example, if this is let us say suppose the order of this group is let us say 3 then what you will get is you will get you know you will get the action of the cube roots of unity on the on the unit disk and well and that will map to something here.

If I draw it, this how it is going to look, as the point therefore, the point is what is going to happen is z naught will be the only point which is fixed by the stabilizer every other point z prime is going to just move and it, it will have this group will act like a group of rotations. Every other point will not have any stabilizer, the upshot of this whole discussion will be that you see give me a point z naught which has finite non trivial stabilizer then I can find a disk like neighborhood of that point z naught where every other point has trivial stabilizer and the action of the group on the stabilizer group on that neighborhood will exactly be exactly be the action of a finite group of rotations of the unit disk about the origin. This is a beautiful thing that comes up how one proves that, so let me sketch that.

There are 2 inputs from complex analysis that we use, you see what I do is well I am going to take a prove let me say proposition or even it is bad to call it a lemma.

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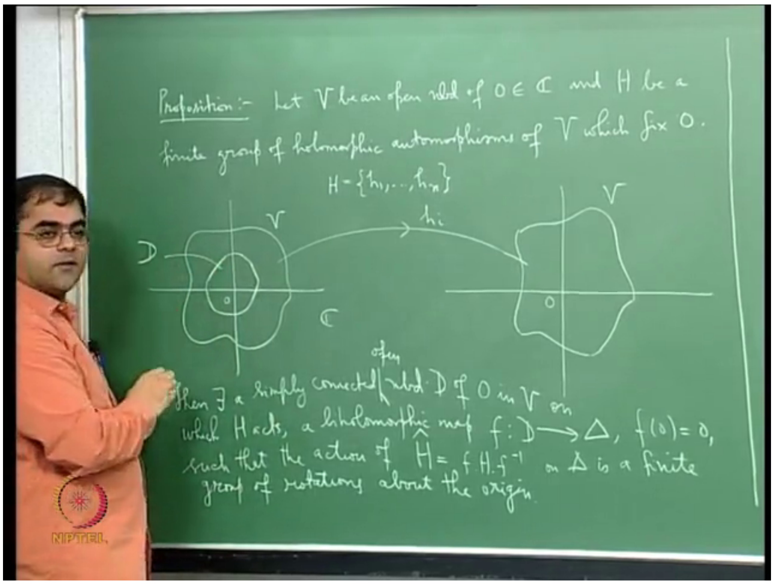

So, what I am going to do is, let V be an open neighborhood of 0 the complex plane and H be a finite group of holomorphic auto morphisms of V which fix (Refer Time:33:29). I am just trying to translate whatever I said into small result, the situation is like this, you see I have, here is my complex plane and I have some there is some neighborhood be of the origin all right and what is happening is that I have a group H. This group is see first of all it consists it is a finite group, I can write the group as h 1 etcetera up to h n right and these each h I is a holomorphic automorphism of V that means, it is a map from V to V, each h I goes from V to V managed to draw something exactly like V. So, each h I maps V back onto itself it is a holomorphic map and of course, it is going to take origin to the origin and there are only finitely many h is right.

Then I am going to claim exactly what I was explaining there then there exists a simply connected open neighborhood of course, D of 0 in V and which on which H acts a biholomorphic map f from D to delta the unit disk such that such that the action of H hat which is defined to be this is action on delta that you get from the action of H on D. So, and of course, you know this by holomorphic map will of course, take 0 to 0 it goes to take 0 to 0 and H hat is just well it is conjugate H by f. It is essentially apply f inverse then apply H then apply f, such that the action of this H hat on delta is a finite group of rotations about the earth. In other words what we are saying is H hat is generally is a cyclic group it is a finite cyclic group generated by rotation by an angle which is 2 pi I 2 pi by n where well n is the order of. In this case I think it is if I take h 1 as identity, where n is order of the group.

How does one prove this, for that there is a little bit of mapping theory that one has to look at, the first thing I wanted to you to recall from complex analysis is the following thing.

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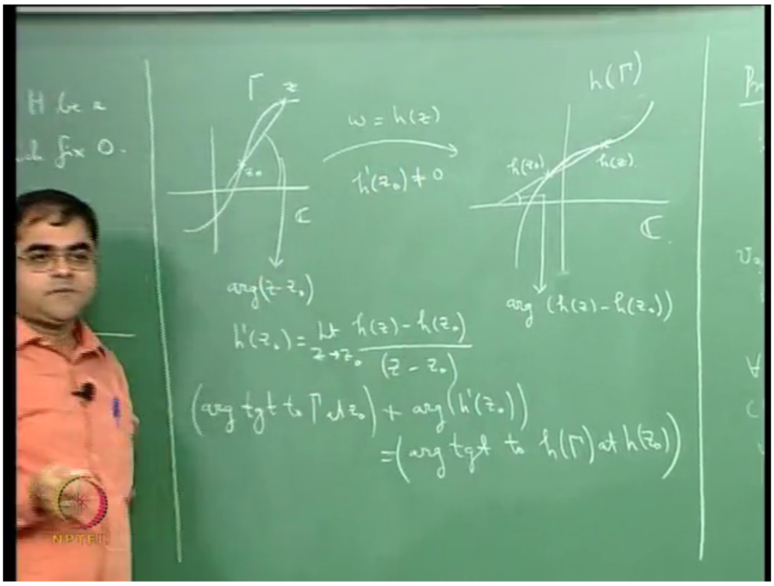

See suppose you have a mapping, this is a complex plane again and I am taking a mapping omega equal to h of z that is going from let us say the z plane to the omega plane. Now you see suppose I take a curve gamma and I h is of course, a holomorphic map right then I will get another curve I get h of gamma I will get another the image of this curve will be another curve there and well if you take a point suppose I take a point z naught and I take other point z then I will get points here h of z naught and I will get a point h of z and well if I see if I join this line from z naught to z all right then this angle is going to be the argument of z minus z naught.

This is the argument of z minus z naught and of course, whenever we say argument you have to one has to read it modulo 2 pi and well if I do the same thing here then this angle is going to be well it is going to be argument of h of z minus h of z naught and now you know this is something that probably you already have come across you know that you know h prime of z naught is by definition limit z tends to z naught of course, I am assuming h is a holomorphic map which means it is differentiable at each point in some domain where this curve is which is an open connected set where this curve is being considered. This is well if I write this down it is h of z minus h of z naught by z minus z naught now you see if I take arguments what I will get is that, I get well as I see if I let z z to tend to z naught then this line will become the tangent to the curve gamma at z naught all right and this line will become the tangent to the image point the tangent to the image curve at the image of the points z naught.

What this will tell you is, that it will tell you argument of the tangent to gamma at z naught plus argument of h dash of z naught is equal to argument of course, when I say argument as a tangent I mean the angle made by the tangent normally argument is defined for a complex number. So, this is abuse of language, but what I mean by that is angle made by the tangent with the x axis with the real axis, and here I will have argument of the tangent to h of gamma at h of z. So, you get this is of course, you know you have to assume that h dash of z naught is not 0 it does of h day I mean you do not want the derivative to vanish I said not because if h dash of z naught is 0 then argument is not defined, if you want this to make sense the derivative should be nonzero, you get this relation just from this and by taking a limit as z tends to z naught.

Now try to apply it to our situation let us try to apply to our situation, in our situation what is happening is, let us, what is the curve I am going to consider.

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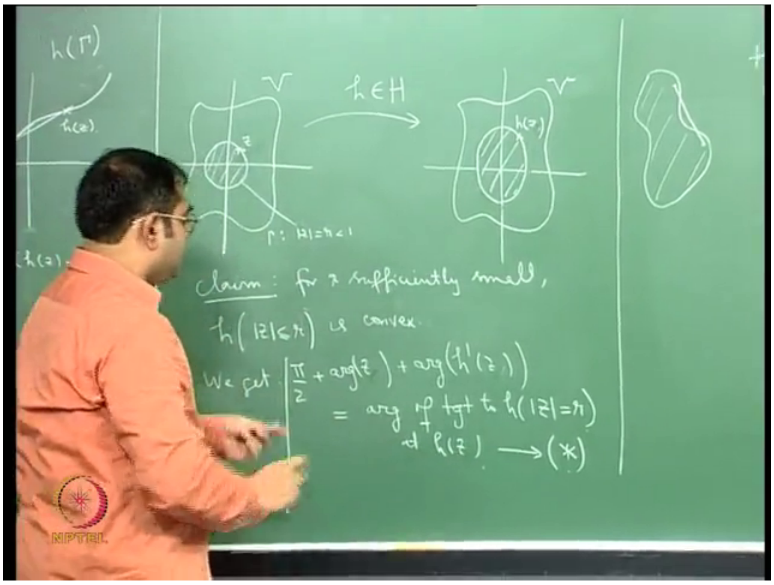

So, you see I have you know select making draw the diagram, I have may not exactly look the way it looks there that does not matter here is my V and here is let us say one of the h from the group capital H and, here is again (Refer Time:42:50), let me draw this. You see if I now take let me now take for the curve let me take a disc centered at the origin of radius say some small r which is less than 1, what I am going to do is I am just going to take, let me just write it like this, this is just mod z is equal to r strictly less than 1.

This is my curve all right, this is my gamma mind you the h that I am going to consider is H in capital H, all these small h are all they are biholomorphic maps therefore, the derivatives cannot vanish. This condition that the derivative does not vanish there is always satisfied and let me try to look at what happens to this curve.

The claim is that the image of this disk of course, it will go to something here what I want to say the important geometric point is that the image is convex. So, what I want to say is that you know when I take the image of this well it is something convex, I be careful drawing it, well. So, you know it is something like this well I am drawing an ellipse because I do not want to draw a circle, but I want to draw anything else that will manifest a look a non convex you know a set is convex if you take 2 points in the set then the line segment joining those 2 points is also in the set.

The point is that h maps this curve onto a convex curve and the region inside this will go to a convex region, the claim is for well of course, for r sufficiently small h of the image under h of this disk mod z less than or equal to r is a convex, I do not need to say v is convex, this is the claim. Now you see if you apply this condition if I take a point z naught here and if I take it is image there it is h of z naught and if I put this condition here see the thing that I will get on left side is see what is the argument of the tangent to the curve at z naught, the argument of the tangent will be you see the tangent will be this tangent of the circle that will be 5 by 2 plus the argument of z naught all right. So, what I will get here is we get pi by 2 plus argument of z naught plus argument of h dash of z naught is equal to argument of tangent to h of mod z is equal to r at h of z naught.

This is what you get all right and you see let us try to look at the condition see the image of this curve will be a curve there and the condition that it encloses a convex region is that the if you take the argument of the tangent that should continuously increase only then you will get a convex you only then you will get enclose a convex region. If a curve encloses a convex region then you are the argument of the tangent has to continuously increase, if the argument of the tangent decreases then the curve I mean it will just get cave inside and you will get a non convex portion. So, you know if you want you can easily look at a simple diagram like this you see if I have a curve well you know.

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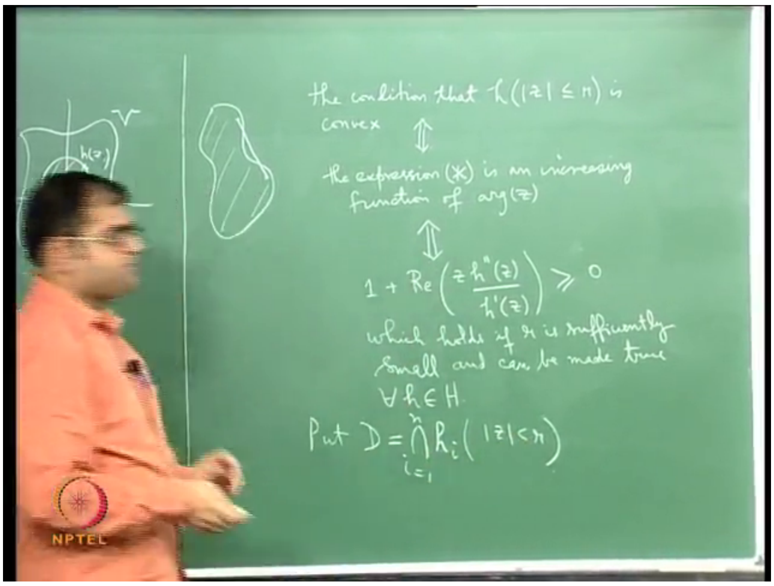

If I take something like this the reason and suppose I look at the region enclosed by this is not going to be convex because you know I have a point here, but the line segment joining these 2 points using inside is not there in my set if I consider it as interior this is not a convex region and see what is happening see the if you look at the argument of a tangent you see it keeps on increasing at some point it decreases and then it again increases.

You see this is what happens, the condition that this curve is convex curve which is the same as saying that they the region enclosed by this curve is convex is the condition that this quantity here should be an increasing function of argument of z naught, as z naught varies on the unit circle. The condition that h of mod z resurrected r is convex is equivalent to pi by 2, let me call this equation as star if you want, the expression star is an increasing function of argument of z that is a condition and well if you write it down, this is an exercise that condition turns out to be the condition that 1 plus Real part of e z h double prime of z by h prime of z is positive. That is literally like you know it is like taking the derivative of this term you are trying to take the derivative of this term with respect to argument of z.

Of course replace z naught by z, let me do that well let me remove the subscript 0 and assume that z is moving on this boundary circle. Then I can remove the subscript z naught and consider all this as you know functions of z right, then the condition is that this right hand side should be an increasing function of argument of z and which means the left hand side should be an increasing function of argument of z and if you write it down it is literally like differentiating this with respective argument of z and this is what will get it is a little bit of simple calculation that you can do.

So, but now notice that you know if you take mind you this is a non zero number and it is a finite complex number for any given z mind you h prime of z it is non zero because the h is a holomorphic automorphism. This is some finite complex number a fix z and then at least I can say that you know this is some bounded quantity all right, but you see if I take z very close to the origin namely if I take this r small r, an epsilon which is sufficiently small. Then the contribution due to this term is going to be less and 1 plus z is certainly going to be greater than 0.

The moral the story is that this is true if you take r sufficiently small, let me write this which holds if r is sufficiently small if you take; that means, you take a sufficiently small disk then the image of that disk here will be a convex region all right and this is for a given h, but there are only finitely many H because capital H is only a finite group therefore, what you can do is you can pick an r which will work for every element of the group h, and can beam can which holds if r sufficiently small and can be made true for all h in H. If I start with say h 1 I will get an r 1, then you take h 2 I will get an r 2 and then you take the minimum of all the r s then it will work all right.

In that case now what we are going to do is, we are going to put you see D to be just you take h i of this mod z less than equal to r the small enough disk and simply take the intersection, what you will get is a convex you will get an open convex neighborhood of the origin and this will be invariant under the group h because I have simply taken images by all the group elements and I am taking the intersection.

Then what will happen is D will be a simply connected neighborhood of the origin which is invariant under h it will be simply connected because it is an open set and it is convex this is the intersection of finitely many convex sets which contain large alright, let me maybe I will erase this diagram.

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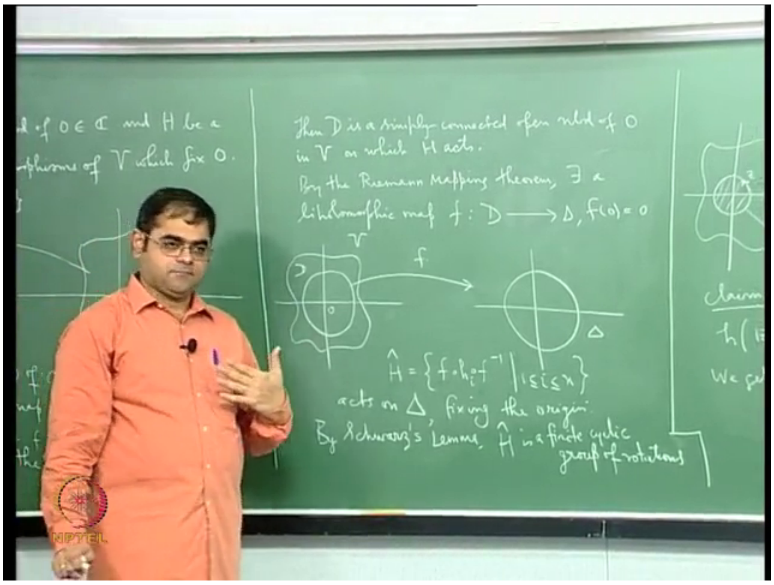

Then D is a simply connected open neighborhood of the origin in V on which H acts maybe I think, I guess one of you are wondering that I am taking the close disc. So, let me get rid of the, if I want something that is open I should take the image of an open set good, sometimes ease dropping also helps fine, that was a small error you see D is simply connected open neighborhood of 0 in V on which he acts, I have gotten hold of this D.

Now, you see we have put the first part of the theorem there is a simply connected openable D of 0 in V on which H acts up to that we have got it alright, then the second, but a clear statement says there is a biholomorphic map of D to delta which takes 0 to 0 now that is directly a consequence of the Riemann mapping theorem.

The Riemann mapping theorem says take any simply connected open subset of the complex plane which is not the whole complex plane then it is by holomorphically equivalent to the unit disk and you can choose the biholomorphic map in such a way that any fixed set point in this can be mapped to 0, by the Riemann mapping theorem there exists a biholomorphic map f from D to delta with f of 0 equal to 0. So, what is happening is you have got hold of well again I should be careful I am trying to draw something that is convex, here is my D and the image of D is unit disk not in this diagram, I will have to draw this diagram..

Anyway let me draw it something like this, I have some D and I have by biholomorphic map of that into the unit disk and I define this group H hat which is just well there is h is a group that is acting here on D now I can make it act on delta because f is after all biholomorphic map. So, what is the definition of my H hat it is just you take any element of h first apply f inverse by that element and then apply f where well where of course, one less or equal to I less than I think. This is acts on delta namely it is a finite group of auto morphisms of the unit disk which fixed the origin.

We have the second important ingredient which comes from complex analysis, it is a following result which is called which is one of the versions of what is called Schwartz lemma. Schwartz lemma tells you taken take any automorphism of the unit disk which fixes the origin it has to be a rotation that is one version of Schwartz lemma which we want to use, each of these maps is an automorphism of unit disk which fixes the origin by Schwartz lemma each one is a rotational and, what you have is, you have a finite group of rotations and therefore, it has to be a cyclic group. The moral of the story is that you get a H hat will be just a cyclic group of rotations, by Schwartz lemma H hat is a finite cyclic group of rotations.

Namely each one is rotation by a certain angle and there are only finitely many angles take the least among them, take the smallest non zero angle by which you have a rotation because this is a group and it is a finite group you will see that that will generate the whole group. Therefore you have succeeded in getting hold of this statement; the picture that emerges is that you see you have a finite group of holomorphic automorphisms of a neighborhood of the origin. Then the way it adds is locally at the point at the origin it is like the action of at this a finite group of rotations on the unit disk, which is isomorphic to dislike neighborhood inside V of the origin.

Therefore, that finishes the proof of that proposition and now we can go back to what we were trying to prove, let me just restate that and finish to proof.

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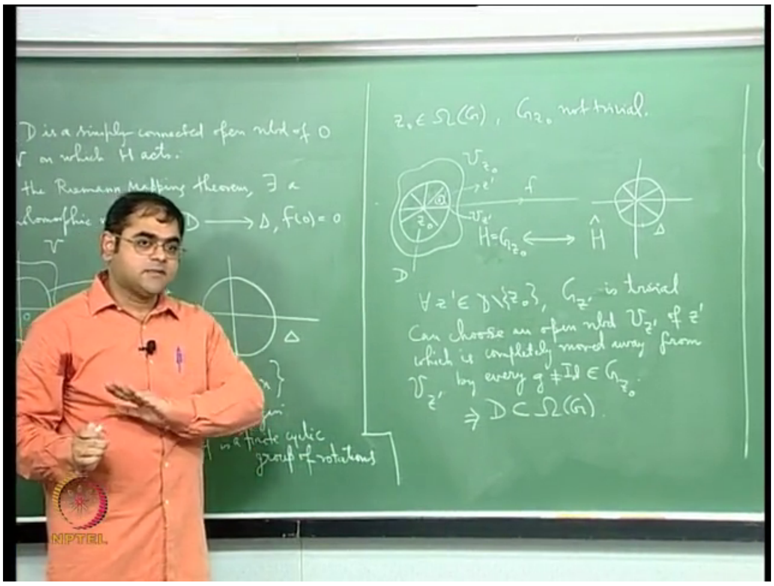

So, you see what we did we took a point z naught in omega of G a point at which G acts properly discontinuously and where the stabilizer subgroup is not trivial then you see by this proposition, my situation is see, what is the situation I have, I have some neighborhood U z naught which is the neighborhood that is given to me because G acts properly discontinuously at z naught and this neighborhood is you know acted upon by this finite stabilizer subgroup and it is pushed away from itself completely by every other element which does not fixed z naught.

By if necessary I can translate this z naught to the origin and I am in this situation and therefore, what I can do is that inside this by that proposition inside this I can find well again problem of trying something that is convex, let me draw something. I will get a convex neighborhood D of the point z naught which is a disk- like neighborhood namely this is this just looks like the unit disk and, the under this map the group H in this case the group H is just G sub z naught that is what is acting on this neighborhood now that corresponds to the group H hat here and the group H hat is a group of rotation, it will act like this.

For example, if it is probably something like this all right, the action of G sub z naught on this we will look like this it will be like you know pieces of the sectors moving just being rotated, this piece, here what happens is let us say this piece goes to the next piece and so on, the same so. In fact, I have not drawn it very nice very neatly maybe I will remove this, so this piece goes to the next piece and this piece goes to this piece, this piece goes to this piece that is how the group acts the group elements and that is going to be the same way in which the group elements are going to hat here. Now it is very clear that if you take a point z prime which is different from z naught then every element of the stabilizer of z naught is certainly going to move it every element of the stabilizer different from the identity is going to move it that tells you that this z prime has to be a stabilizer.

Let me write that down for all set prime belonging to D minus z naught G z prime is trivial because it is just going to move around the point in some sense and the stabilizer is trivial and of course, you know I can choose a small enough neighborhood of this z prime which is going to be completely moved away from itself by any non trivial element of this stabilizer. I can, so this point is z prime and this neighborhood is U sub z prime and this neighborhood U sub z prime will satisfy the condition will help you to verify the condition that G is acting properly discontinuously at z prime with z prime having no stabilizer can choose z prime such that can choose an open neighborhood use at prime of z prime which is completely moved away from U z prime by every G naught equal to identity in G z.

This implies that this neighborhood D is contained in omega of G, that finishes the proof, let me recall what we are saying, what we are saying is you look at the set of points in the external complex plane where the group acts properly discontinuously. Then the set of points where the group is are going to act without fixed points that is an open subset because for every such point you can find the whole neighborhood where the group is going to act properly discontinuously and without fixed points. On the other hand you take a point where the group has a fixed point; that means, there are the stabilizer is nontrivial then also you continue to find a neighborhood of that the disk like neighborhood of that point which is such that every other point except this one is going to be a point where the group acts without fixed points and properly discontinuously.

So, and in this way what you know what you get extra is the set of points where a G acts with finite stabilizes is discreet because you have separated 2 such things every such point where G acts with a finite nontrivial stabilizer is surrounded by whole a disk like neighborhood where G acts with trivial stabilizer, the another of short of this is the set of points where G acts with finite nontrivial stabilizer is a discrete subsets of the external complex plane.

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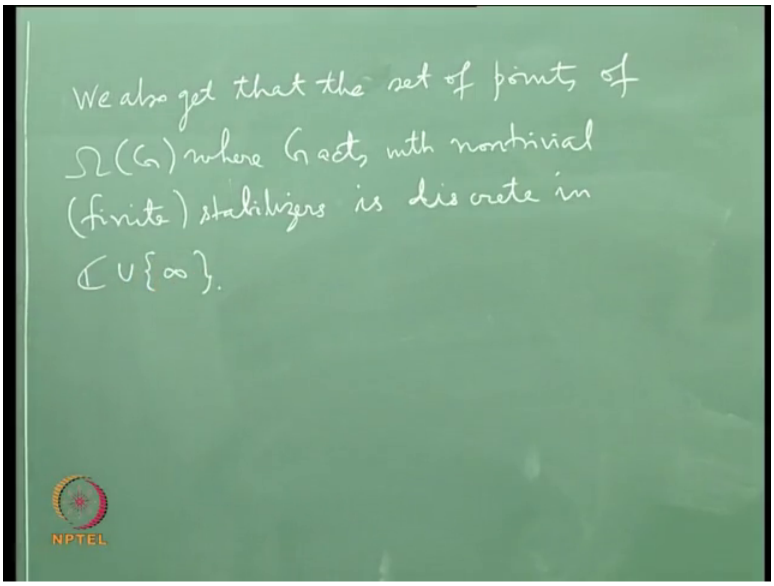

Let me write that down we also get that the set of points of mega G where G acts with nontrivial finite stabilizers is discrete in C U infinity. The model of the story is that the action when the group acts properly discontinuously it is and there is at least one point where it acts properly discontinuously namely you take a claim in group then it has ah very good properties. So, you precisely know how this structure of the action looks at a point which has nontrivial stabilizer which is finite stabilizer. This will be helpful for us when a in the in the forthcoming lectures to be able to give a quotients give to mode of by such a group and give a Riemann surface structure on the quotient, I will stop you .