An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1 -dimensional Tori and Elliptic Curves

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Lecture - 25
Galois Coverings are precisely Quotients by Properly Discontinuous Free Actions

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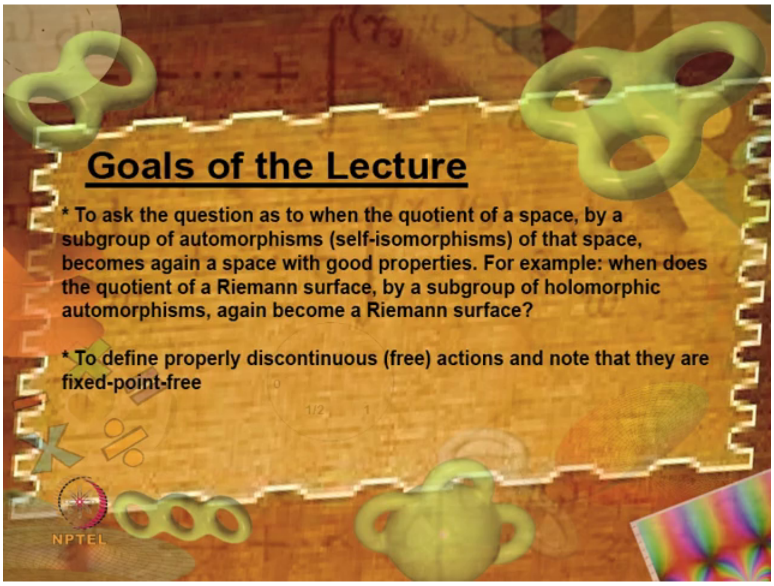

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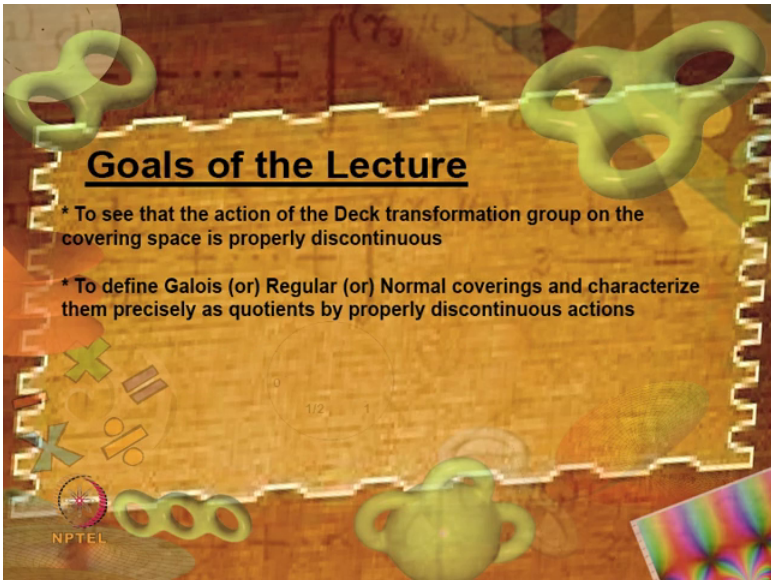

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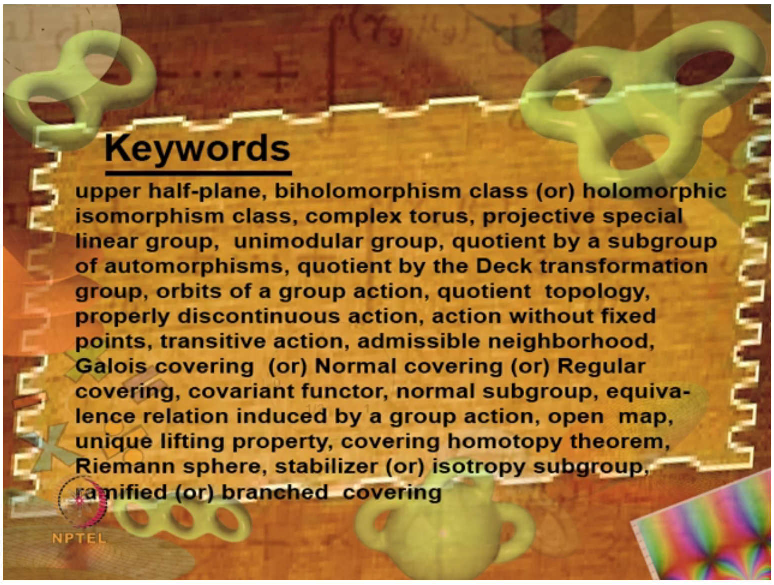

Well, last time you see we were looking at the statement that the set of holomorphic isomorphism classes of complex tori is in a natural bijection with the orbits of P S L 2 with integer coefficients on the upper half plane and what we need to do next is to show that this set of orbits, namely U mod P S L 2 z is actually a Riemann surface and. In fact, the claim is that this Riemann surface is just bi holomorphic to the complex plane we need to prove that. And therefore, first of all we need to see how the quotient U mod P S L 2 Z even becomes the Riemann surface.

So, you see then of course, it is another matter to prove that this Riemann surface is bi holomorphic to C; there it is isomorphic to C. So, first of all we need to know how you how you need to know how you are able to get a Riemann surface structure on U mod P S L 2 Z alright. So, which means roughly the problem is like this the problem is u itself is Riemann surface after all it is an open subset of the complex planes. So, it is a Riemann surface any open subset of a Riemann surface is also natural a Riemann surface, and P S L 2 Z is a group of you know holomorphic automorphisms of U ok.

Of course, you know all possible automorphism of u are Mobius transformations with real entries alright and then Mobius transformations with integer entries if you take with elements of P S L 2 Z they are also going to be automorphisms of U, and the problems is to go u mod P S L 2 Z. So, the general problem us to take a Riemann surface and then go modulo a certain subgroup of holomorphic automorphisms of that surface and to see where you can get Riemann surface structure on the quotient ok. Obviously, you cannot expect this to happen for an arbitrary sub group of holomorphic automorphisms. So, it will happen first subgroups with special properties. So, what one wants to do is to try to single out those properties, what are the crucial properties of the of a sub group of holomorphic automorphisms of a Riemann surface, so that when you divide by the subgroup the quotient again becomes the Riemann surface.

So, first of all let us look at. So, what I am going to do is try to look at the situation for covering spaces, that is I am going to look at the situation say for example, even the universal cover if you take the universal covering space then universal covering space modulo the deck transformation group is going to give you the space below, the space that it is that this universal covering space is actually covering. So, that is the starting point.

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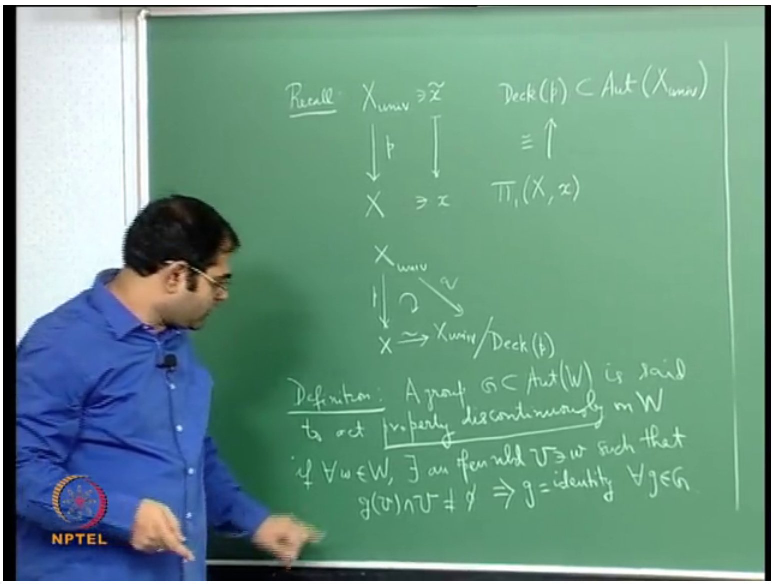

So, well so let us recall see if you have. So, let me go back into for example, even the topological category if you want, but of course whatever I am going to say will also work in the holomorphic category. So, suppose X is there topological space. Then of course, we are of course not working with arbitrary topological spaces which are you know housed of arc wise connected locally arc wise connected, locally simply connected. So, you take a topological space like that. And then you have the universal covering and you have the you have this universal covering space, this is something that we constructed and what we noticed here was two things, we noticed that if you fix a point X here and you fix a point above it x tilde and which means p maps x tilde to X. Then we noticed that the fundamental group of the base based at the point small x can be naturally identified with the deck transformation group of p, which is subgroup of automorphisms of T universal covering spaces and well. And we further also noticed that you know if you take this quotient is simply the universal covering modulo this group namely the group of deck transformation

So, In fact let me also write that X to X sub univ to X, and if I write X sub univ to X sub univ modulo deck of p. This is p, and if I call this map as this is a map natural map the mind you the deck transformation group is a subgroup of automorphisms the universal covering and it is a group that acts on this and this is just the orbits the space of orbits. And this is the natural map sending every point to it is orbit under the action of this group.

Then I told you that if you call this map as let us say q. Then there is a there is a natural identification of X with this in such a way that this it is. So, you can identify X with this with this set given the quotient topology; namely the topology that makes q into a continuous map alright and we have already seen this. Now, one wants to look at a slightly more general situation. So, here is the here is the first example, you have a you have a space you have this subgroup of automorphisms of this space and then you are able to divide by that to get. So, that the quotient map becomes this universal covering.

Now, more generally what can you do? So, well so it happens that the reason we are able to get you are able to divide by this group and get this as the nice covering has got to do with the way in which the deck transformation groups acts on this. So, well let me make a definition. So, a group G of aut; let me take a topological space let me call it as say Y or let me even call it as W. So, W is a topological space and G is a sub group of the automorphisms of W.

Of course, you know if I take was a topological space then these all homeomorphisms, self homeomorphisms and G is a sub group of self homeomorphisms I could also take W to be for example, if you want Riemann surface, and then these then the automorphisms that I am going to consider are going to be automatically holomorphic automorphisms and G will be a sub group of holomorphic automorphisms. So, really the argument does not depend on whether I am working in the topological category or in the holomorphic category ok.

So, you take a group like this, such group is said to act properly discontinuously on W. So, this is the important notion properly discontinuously on W. if So, maybe I should if for every small w in capital W there exists an open neighborhood u containing w such that you know G of U intersection U is non empty implies G is equal to identity for all G in G. So, please try to understand this definition.

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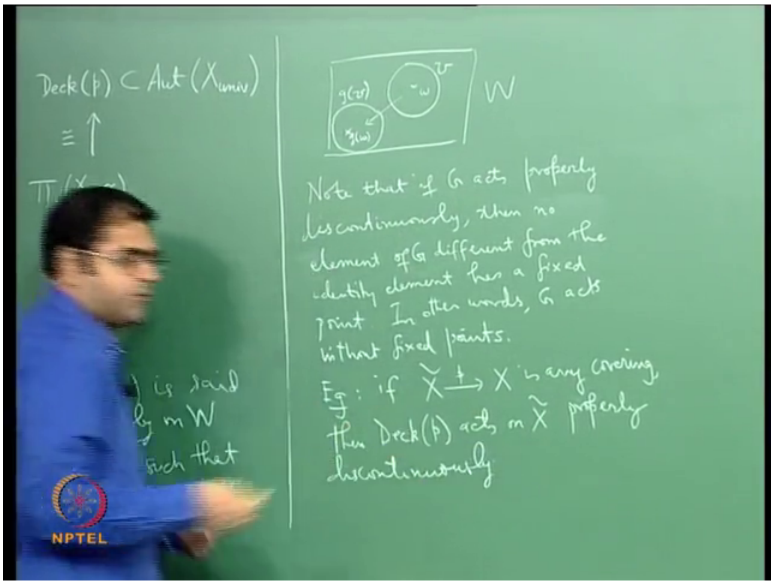

So, you see if I if I try to draw a picture that picture is like this. So, here is my here is my W, and you see and am and I am looking at elements of small g belonging to capital G these are all automorphisms and certainly they are homeomorphisms ok.

Now, you take you take a point small w and take a neighborhood. if I a apply if I apply g to this you can expect that I will get obviously, I will get W will be taken to g of w and if this neighborhood is U it will go to a neighborhood g of U So, g of U will be just a neighborhood of g of w. And of course, you know g will be induce an it will induce homeomorphism of u with g of u, because g is a homeomorphism and homemorphism restricted to an open set is going to give you a homeomorphism on to the image of that open set.

So, the question is I mean the point is that when you. So, what this g does is that it moves this neighborhood of w to a neighborhood of g dot w, and the condition is for each w I should find a neighborhood, such that no matter what g I take that g should displace U completely away from itself. If I take a that is; if I take g different from the identity of course, if g is identity is not going to do anything if you take an element g an element of the group which is not that identity that is you take non trivial automorphism then that automorphism any non trivial automorphism from your group has to move U completely away from itself, that is there should be no intersection between U and g.

So, the condition is you must find such a neighborhood U that will work for every g. That is there is a neighborhood u of w which is pushed by every non trivial element of g completely away from u that is the condition. So, this is the condition for properly discontinuous action and well if you look at covering space then the deck transformation group does act in this way. So, the first thing I want to say is that the movement if g acts properly discontinuously then the first inference that one has to make is that g has a fixed points.

Note that if G acts properly discontinuously, then g has no fixed points. So, what I mean is then no element of G different from the identity the identity element of capital g different from the identity element has a fixed point it is obvious, because you see if an elements small g belonging to capital G has a fixed point. Then that is going to be a point w such that g of w is equal to w, but you see w is in u you take any neighborhood u, w is in u g of w in g of u, but g of w is equal to w. Therefore, u and g of u will intersect certainly it will contain w and therefore if I take a neighborhood like this that cannot happen. So, if it acts properly discontinuously then G cannot have any fixed points say in other words this definition of properly discontinuous is going to imply a stronger than assuming that G has no fixed points ok.

So, in other words let me write that in other words G acts without fixed points, there are no fixed points. So, what is the standard example, if X tilde to X p is any covering then the deck transformation group of that covering acts on X tilde properly discontinuously. If you take a any covering then you know the deck transformation group is a sub group of automorphisms of the covering the space above and that will act properly discontinuous this is just because of the presence of the admissible neighborhoods.

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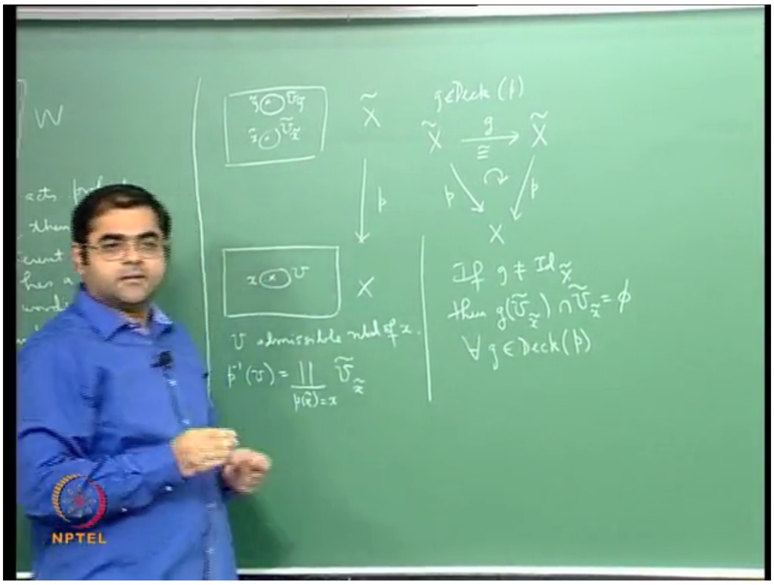

See because for a movement you know if I draw a diagram, so here is my X tilde. So, here is my covering space, here is my p and here is my X, suppose I have something like this well given any you know give me a point here, let me call this as small x tilde, you know then if you take it is image here just small x. So, small x tilde goes to small x, then this x has a has an admissible neighborhood U. So, that is this U admissible neighborhood of x if you take this, then you know that if you take p inverse of U, the inverse image it is a disjoint union of admissible neigh it is a disjoint union of open sets and each of which p maps homeomorphically on to U. So, p inverse U is going to look like you know for every.

So, p inverse U is going to be some U alpha, maybe I can use U tilde alpha. And well of course, this is this is indexed by all possible x tildes where p of x tilde is equal to x. So, you take any point x tilde that lies above x; namely any point x tilde that goes to x under p, then there is then surrounding this point I have neighborhood U tilde sub x tilde which p will map homeomorphically onto this U. And this will happen for every point if I take some other point then I am going a suppose I take a point y tilde then I am going to end up with another I am going to just get U tilde sub y tilde and well p is going to map this U tilde sub y tilde homeomorphically onto U.

And now you see for the point x tilde if I take this neighborhood U sub x tilde which is one of the pieces in the inverse image of an admissible neighborhood admissible neighborhood of the it is image below of the then this will satisfy the condition of this will give you the condition that you require for a properly discontinuous action. So, you see; let me explain so you see suppose I take suppose deck transformation. So, deck transformation g is an element of the deck transformation. So, what does it mean it means you see I have I have a map g, g is a homeomorphism of x tilde onto x tilde of course, if I am in the holomorphic categories holomorphic isomorphism such that if preserves the fibers of the covering. That means, if I apply g and then apply the projection I will still get back the projection. Of course another way of saying this is that g is a lift of p which is an automorphism of x tilde it is a lift of p. And well you see the fact is if I apply g to if I take the deck transformation and if I apply g to this neighborhood, suppose I choose x tilde and I choose this neighborhood then g dot that neighborhood will not intersect this, if g is not the identity.

So, if g is not equal to identity this identity transformation of x tilde then you see g of U sub U tilde sub x tilde this neighborhood this the this will completely different this will not intersect U tilde sub x tilde, this will be this will be a null set for and this will hold for every g in deck in the deck transformation of course g not equal to identity and that is obvious, because you see g will move x tilde to because g is going to preserve fibers x tilde lies over x. So, g of x tilde will be another point which will again lie over x, and for that and g of u tilde of x tilde will be also a neighborhood of that g of x tilde alright and that will also be mapped homeomorphically onto U.

So, it is in this inverse image. So, it has to be, but you know all these inverse images they are all disjoint. So, therefore, this cannot intersect alright. So, the moral of the story is that the deck transformation group if you take any covering space not u not universal covering take any covering space you know. of course universal covering is special case when the space above is actually simply connected, but you take simply any covering then the deck transformation group is always going act properly discontinuously fine.

So, this is a standard example of well properly discontinuous action alright. Now we come to trying to generalize the following we go back to the situation, see if you take the case of the universal covering I am able to take the top space, I am able to divide by the deck transformation group and I am able to get the covering. So, of course mind you with these deck transformation group is of course, this fundamental group of this guy of this space x below. Now you can ask this question take any covering space the deck transformation group of this cover acts properly discontinuously on the top space can you divide by that and can you get this cover.

So, there is an obvious answer, you know that if of course, this will not always not happen and there is an there is an obvious restriction. For example, if I have to divide by this and get x then x should be the set of orbits of the deck transformation group on x tilde. And that means, that will mean in particular that deck transformation group should act transitively on the fibers. So, that is already a restriction you cannot expect to take any covering space like this, even though the deck transformation groups acts properly discontinuously on x tilde you cannot expect x tilde modulo the deck transformation group to give x.

So, there is something slightly more that you assume and this is where I mean this is the beginning point of Galois Theory of Coverings. So, you have to assume as special condition; namely the condition that the covering is Galois other words that are used for that are normal cover or regular cover.

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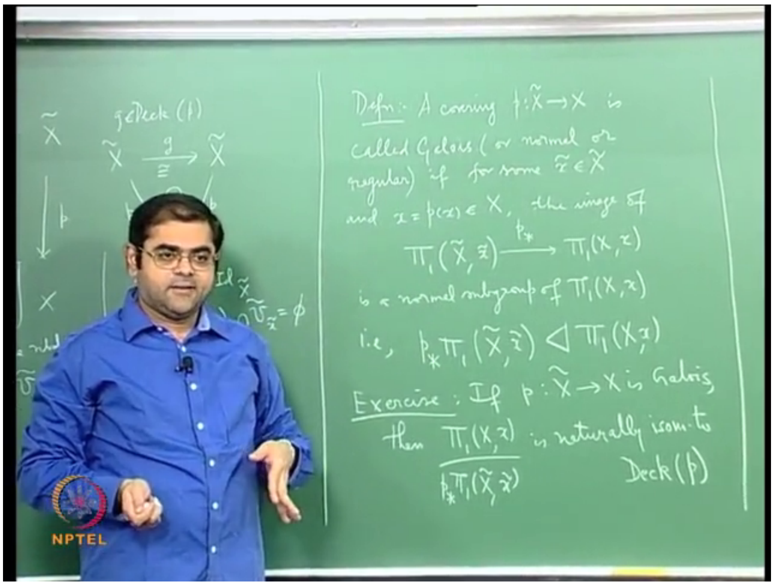

So, I will make this definition a covering a covering p from X tilde to X is called is called a Galois, or normal, or regular. These are other words that often used. A covering is called Galois if given if for some x tilde belonging to x tilde, small x tilde in capital X tilde and it is image x equal to p of x is small x, you see I have the fundamental group of x tilde based at small x tilde. And because you know the formation of the fundamental group is a covariant; what I will get a homeomorphism, I will get a group homeomorphism p lower star which is going to go from the fundamental group above to the fundamental group below, which is a very simple map it is just take a loops it will takes a loop above. And then composes projects it down you get a loop below and then you do this mod f e p homotopy and you get this map and it is a group homeomorphism.

So, what I want is see this is a group when I take the image the image is only a sub group. So, the condition is that the image should be a normal sub group. So, let me complete this sentence a covering space from x tilde to x is called Galois or normal or regular if for some point above and it is image below the image of this is a normal sub group of the target. So, I want to write it in symbols. So, I will write p lower star pi 1 x capital X tilde small x tilde is a normal sub group of pi one capital X comma small x. So, this is the condition.

So, normally we use this triangle symbol to normally will sometimes use a lesser than symbol to say it is a subgroup and then we use a triangle symbol to say that is normal sub group. So, this is the condition the condition is that the image of the fundamental group above in the fundamental group below is a normal subgroup then something nice happens.

So, once you have a Galois covering then what happens is? So, it is a very simple exercise to check that this group divided by this normal subgroup is going to give you a quotient group and that quotient group is none other than the deck transformation group, you see look at the situation when this is a universal cover, when you have a universal cover then this is trivial. So, it is image is trivial sub group trivial sub group is a normal sub group, the quotient is just this fundamental group below and that is isomorphic to the deck transformation group so in this case also.

So, it is an exercise which is very easy exercise to do is, if p from X tilde to X is Galois then the fundamental group below by the image of the fundamental group above which is normal subgroup. So, I can divide by that this is naturally isomorphic to the deck transformation. So, the fundamental group below by the normal group which is the image of fundamental group above, this quotient group is naturally isomorphic to the deck transformation group ok.

Now, what it is it? So, what how does this definition help, this definition helps in the following that in this case that is if you take a Galois cover and take the space above and divide it by the deck transformation group above the what you get is exactly the space below.

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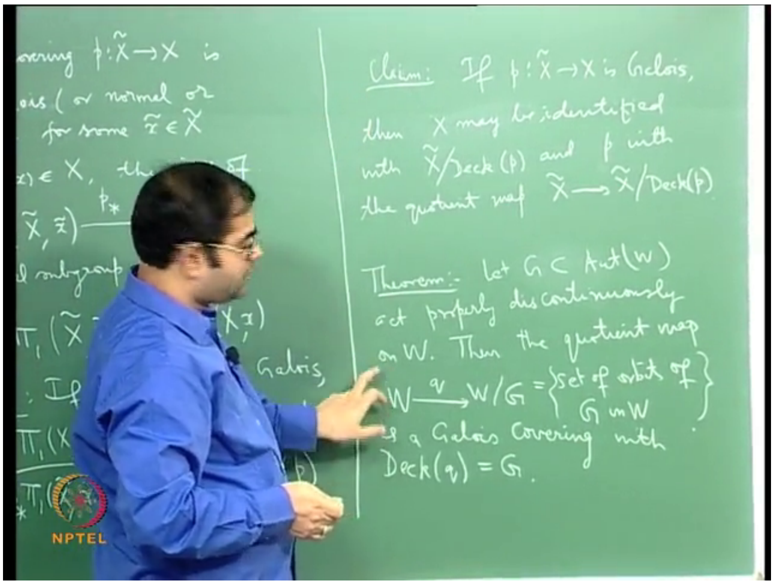

So, the fact is the claim is claim is if p from X tilde to X is Galois, then p may be So, I should write then X may be identified with X tilde modulo deck p and p with the quotient map X tilde to X tilde modulo deck p. So, the moral of the story is that if you have a Galois cover you take the deck transformation group above, then you divide by the deck transformation group what you get is the space below. This is a generalization of the case when if you take a universal cover you take and you divide the universal covering space by the deck transformation group which is the fundamental group below and you get the space below. So, this is the more general case. So, actually the Galois area of covering spaces comes by studying these Galois covers well. So, to prove this claim let us let me state and prove more general theorems.

So, here is theorem let w be; you have to make some basic assumption. So, let us assume that it is you know it is a topological space satisfies all our usual conditions; namely you know (Refer Time: 30:41) you know arc wise connected locally arc wise connected locally simply connected. Let us put these blanket assumption though all of them may not be needed for the proof.

So, let G sub group of automorphism with w act properly discontinuously on W, then the quotient map W q to W mod G, so W mod G is set of orbits of G of in W, then this canon then this quotient map is a Galois cover, it is a Galois covering with Galois with deck transformation group, precisely G with deck q equal to G. So, this is the more general theorem. So, if group acts properly discontinuously on the space W then the map natural quotient map from W to W mod G is actually a Galois covering and the deck transformation of this group of this covering is exactly G. So, let us prove this theorem and also try to get this claim this claim will follow as a corrolorry to this. So, let us say to prove that theorem it is pretty easy to prove.

So, you see so I have.

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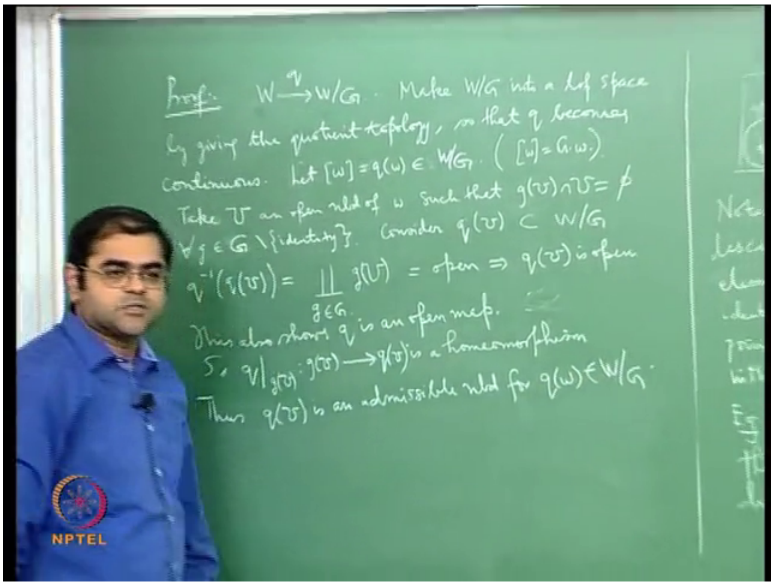

So, proof so you see my situation is. So, I have I have the map W to W mod G I have q. Now what I am going to do you know if you look at the analogy of this covering space situation the admissible neighborhood for a point below that is one it is exactly one of it is inverse images that gave you the open set that is required for the for verifying the condition of properly discontinuous action. So, conversely if you think given any point, if you have that neighborhood which verifies the condition for a properly discontinuous action. Then it is the image of that neighborhood below you should expect that is going to be the admissible neighborhood of the point below of the image of this point below, that is a natural way to think and in fact that is the case.

So, first of all well you W mod G is not even topological space. So, make it a topological space by giving the topol the quotient topology. So, make W mod G into a topological space by giving the quotient topology. So, that q becomes continuous. So, you know what the quotient topology is as set in W mod G is open if and only if it is inverse image under q is an open subset of W. And therefore, by definition the quotient topology is given in such a way that q becomes the continuous map.

o, now at least at least this is a topological space and this is a continuous map. Now what one does is fix or even pick let W be a point. So, this is q of W be a point in W mod G. So, w small w is a point in capital W and I am taking it is image here, if I take it is image here I will get an inequivalence class and I am calling it as bracket w may be and what is this bracket w this bracket w is just the orbit of w under G. So, bracket w is just G dot w, it is just the orbit of w under the action of G and of course I put this flower bracket because in that orbit if I take some other element W prime and take it is orbit I will get back the same orbit.

So, I can replace w by w prime inside this square bracket it is an equivalence to elements being an orbit is an equivalence relation and it is not that equivalence relation that we are going. So, take this now choose w as to. So, there is see since g acts properly discontinuously on w there is an open neighborhood U of this point w which is completely displaced by every non trivial element of g. So, choose U, or let me say take U an open neighborhood of w such that you know g U, g of U intersection U is empty for all G in the deck transforma in the capital G with capital G minus identity ok.

So, because we are given that the group capital G acts on capital W properly discontinuously given a point small w in capital W, there is this U such that g u intersection U is empty for g is not equal to identity take that. And now consider g of U sorry not g of U, q of U which is the subset of W mod G, I am simply taking it is image under q.

Now, the claim is that this is the open and this is the admissible neighborhood for this point q of W. See if I take q inverse of q of U, if I calculate q inverse of U then what I will get is I will get all translates of U by elements of G. So, what I will get this is I will just get various G dot U or G of U translates by G. And this will be a disjoint union over G in G this is what I will get. See if I have point above if I take it is image below and if I take the inverse image what I will get I will get all translates of that point by the group, this not only holds for a point it holds for a set also if I take a set above that is a set in W, if I take it is image q of s. And if I try to take inverse image then I will simply get all translates of s by various elements of G.

So, I started with the u above I take I take it is image below if I now take the inverse image I will get all translates of U, but you see all these various translates of U they will not intersect U, because for G different from identity I will get this disjoint union exactly one of this; namely for G equal to identity I will get U, for G not equal to identity I am going to get a G of U which is not going to intersect U, and mind you if you take G if you take g 1 and g 2 different then g 1 u and g 2 cannot intersect, because if g 1 u and g 2 u intersect you can prove that g 2 inverse g 1 u will intersect u. So, this is actually a disjoint union alright in any case since u is open if g is open is anyway union of open sets.

So, this is open and this will tell you that q of u is open, because that is the to quotient topology. A set below is open if and only if it is image is open above alright. So, q of u is certainly an open neighborhood. And you see it is obvious that see from this is an another thing that this argument tells you see forget instead of U, if I had taken any other open neighborhood. Let us say I took an open neighborhood V, if I push it down take that is take U of V and then I take q inverse of V, I will simply get union of translates of V by G, that still going to be an open set only thing is it might intersect, because I took U this was a disjoint set union, but if instead of U I took some other open neighborhood V what I will get here is not a disjoint union I will get a union of G of v.

So, what it means is in other words it tells you that q is an open map, this is also shows q is an open map and you see now you see if I take if I restrict that q to g of U then q restricted to g of u will go exactly to q u, it will be a bijective continuous map which is also open. So, it is the homeomorphism. So, q restricted to g dot U I mean g of u from g of U to U is a homeomorphism. So, this will tell you that this open set U I may be I should put not this is not U this should be q of U this is a homeomorphism therefore, what it tells you is that this q of U is an admissible neighborhood for the point q of small w. So, let me rub this off.

So, I can write something more, thus q of U is an admissible neighborhood for q of w in W mod G. So, what has happened is that every for every point you have found an admissible neighborhood. Therefore, this is covering space, the only thing that I will have to check is that the deck transformation group of the covering is exactly G. And I will also check that the deck transformation group is you know fundamental group below divided by image of the fundamental group above that also can be checked. So, how will you prove that the jegged the deck transformation group is exactly G it is very simple. So, what you do is; so let me let me do it here. So, that I have more space to write.

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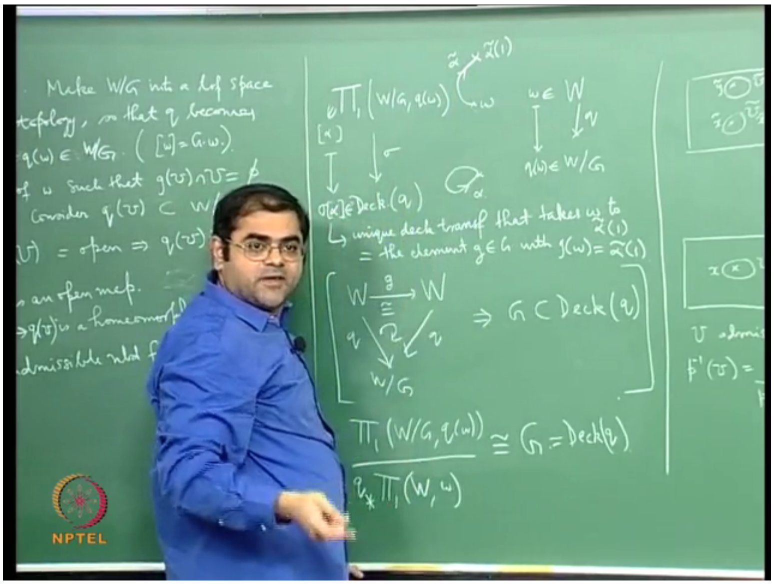

So, you see. So, let me fix some points. So, I have q W to W mod g well I will fix this point W and then I will fix this point q of W, now I will take the fundamental group at of the base at q of w and from this I am going to define the homo I am going to define a homomorphism into the into the deck transformation group of q, I am going to define this see. So, the first remark I want to make is that take any element small g belonging to capital G, any element by definition is going to be an automorphism of W which going to the respect the fibers of U because this is just going modulo G.

So, what I am trying to say is; so let me write that here if you take. So, here is my G if you take. So, here is my G this is an automorphism of W and I claim that W if I take q to W mod G if I put q here this diagram commutes, that is because you know if I take the point x here and if I it will go the point g dot x, and x and g dot x are in the same orbit.

So, they will go the same point below, because after all w mod g is just the set of orbits therefore, by definition every element of G is a deck transformation group. So, you see. So, this implies you know g is contained in the deck transformation group g is contained in the deck transformation group that is very clear every element of g is automatically deck transformation by definition, you will have to we would like to say that this is these are all the deck transformations.

And well, if you go back and look at the way these deck transformations were defined. So, how was this deck transformation group defined well you see what happened there was this point q of w, if I take an element here this is going to be a loop alpha it is going to be a homo topic class of loop alpha at the point q of w and how what is the deck transformation of I defined above.

Well, you see here is my w I take the lift of alpha start starting at small w. So, I will get a unique lift alpha tilde and I take it is empty end point namely alpha tilde of 1, now that is a point that also going to lie over alpha and you see the since these two points. So, you know 2, I want make two remarks. So, this map sends alpha 2 sigma of alpha what is the sigma of alpha this is the unique. So, let me put belongs to symbol here this is belongs here. So, this is the unique deck transformation that takes w to alpha tilde of 1, see try to remember recall two deck transformations if they coincide at one point they coincide at every point that is because deck transformations are lifts of the covering projection and the covering has a the covering projection has a unique lifting property.

So, to specify if you want to kneel down a deck transformation you have to just say it takes this point or that point that is it, you cannot have more than one deck transformation that fixes that carries a particular point to some other particular point. So, I get I take this unique deck transformation which takes w to alpha tilde of 1. And of course you know if I replace alpha by a homo topic loop then you see this homo if alpha is we place by homo topic loop alpha prime then the lift will also a be can be replace by a homo topic loop because the covering homo topic theorem says that you can lift homo topics.

Therefore, it only depends on your alpha the homo topic classes of alpha so, but you see there is certainly deck transformation that carries w to alpha tilde of 1 what is it see since both of them go to a same point q of w they have to differ by a group of element by a by an element of by an elements small g belonging to capital g. And therefore, there is a small g which takes w to alpha tilde of 1. So, that is going to be a deck transformation that takes w to alpha tilde of 1 and that has to be this. So, this has to be equal to the element small g belonging to G with g alpha of w is equal to alpha tilde of 1. An element like this exists because these two are in the fiber over q of w which is the whole orbit.

So, that it has to be that deck transformation groups that element of the deck transformation group. So, the moral of story is that you know the image of this the image of this is exactly G. And what is in the kernel? The kernel will be precisely those alphas which are liftable to alpha tildes such that alpha tilde of 1 is equal to w. So, what is the kernel of this map. So, the image of this you can check that this is a group homeomorphism, the image of this group homeomorphism is precisely the deck transformation group G by what I have told you alright and the kernel what is the kernel suppose alpha is in the kernel that means, alpha is liftable alpha comes from a loop above, but if alpha comes from a loop above then this alpha tilde has to be a loop centered at based at w, that means alpha tilde of one has to be w, that means the deck transformation has to be a trivial.

So, the kernel of this is precisely image of the fundamental group above. So, if you look at this you can see that you know the fundamental group below based at q of w divided by at least as cosec space q lower star of the fundamental group above based at w is exactly isomorphic to G. So, of course, you will have to do little bit of calculation and it is very easy to check that this is a normal sub group, it is a very simple calculation to check that this is a normal subgroup you can check that. So, this is actually this quotient group is actually G that which is exactly the deck exactly the deck transformation group.

And let me also add any deck transformation if I take it is image under w I am going to get some point then I can join it by an alpha tilde and project that alpha tilde to an alpha and that deck transformation will be image of this alpha. So, you see this sigma is also a surjection to deck and deck is and deck is exactly G that is what I am trying to say I am. So, here I just said that G is a sub of deck; I am also trying to say that every element of deck is also an element of G.

So, let me again repeat it you take an element of deck, take w it will it will that element of deck will move w to some g w then I connect this by a path because after all it is path is connected then that path if I push it down I will get a path, I will get a loop below and the deck transformation induced by this by this map may be precisely the unique deck transformation that takes this to that. So, you see. So, it is actually equality this is actually equality that is how you get this equality. So, the moral of the story is that whenever a group acts properly discontinuously on W then W to W mod G is Galois cover deck transformation group is exactly G ok.

Now, the point I want to make is. So, this gives you a very clear picture about how you get quotients in this case, but let us look at the kind of quotient we are trying we are interested in. So, let me let me also add one more thing. See notice that the movement W is Riemann surface, then because this is the covering W mod G also becomes the Riemann surface, because after all this is whenever you have covering and when if the base is the Riemann surface then you can give unique Riemann surface structure to the top, if the top is Riemann surface you can give Riemann surface structure to the base; so if W is Riemann surface then this also becomes the Riemann surface and this becomes the nice covering.

So, the moral of the story you take any Riemann surface take a subgroup of holomorphic automorphism of the Riemann surface and assume that the sub group acts properly discontinuously, then Riemann surface modulo that sub group that becomes the Riemann this natural quotient map becomes a holomorphic covering and the deck transformation it becomes. In fact, the Galois covering it becomes a regular covering Galois covering and the deck transformation group is actually this group G that is acting ok.

So, now you know how to divide by sub group of holomorphic automorphisms which acts properly discontinuous, but there is still a problem here what is happening is notice that this definition of properly discontinuous action already supposes that G acts without fixed points, but that is not that is not enough for our purposes. So, you see let me go back to what we are to the quotient problem that we are actually interested in. So, the quotient problem that we are actually interested in is.

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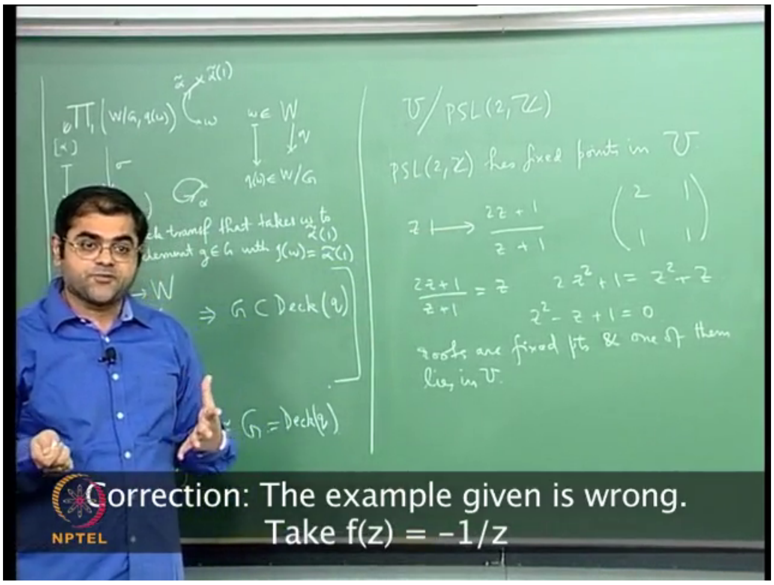

You see we are trying to look at U mod P S L 2 C, this what we are trying to look at this is the this is the upper half plane and P S L 2 C is Mobius transformation with integer entries and we are trying to look at the orbits here and we are trying to I mean our claim is that this is the Riemann surface and this is bi holomorphic to the complex plane. So, what is the problem in this case?

See the first problem is P S L 2 does not act without fixed point on U. So, you cannot apply this theorem. P S L 2 does not act properly discontinuously in the sense of this theorem because it help it will have fixed points see for example, P S L 2 has fixed points in U. See for example, you know it is very easy to write down. So, you know maybe I can take if I take Z going to let say 2 Z plus 1 by 2 by Z plus 1 something like this, if I take this then this is if I write it in matrix form the matrix is 2 1 1 1 it is determinant is 2 minus 1 is 1.

So, it is an element of P S L 2 and it has integer entry. So, it is an element of P S L 2 Z, and what are it is fixed points well 2 Z plus 1 by z plus 1 if I solve if I equate to Z and solve it, I will end up with well I will get 2 Z square plus one is equal to Z square plus Z. So, I will end up with Z square minus Z plus 1 equal to 0. And well, I can adjust these numbers well in any case this is the quadratic equation if I solve for it I will get 2 the quotients are real.

So, I am going to get two complex numbers which are conjugate one is line the upper half plane, the other is it is conjugate going to line the lower half plane and these are going to be fixed points for this element of P S L 2. So, moral of the story so let me write this roots or fixed points and one of them lies in U, in fact you know we have seen an automorphism of U is elliptic if and only if it fixes the point of U.

So, these are elliptic elements. So, P S L 2 has elliptic elements. So, there are fixed points. So, it is not going to add properly discontinuously in that sense because in that sense because properly discontinuous action in that sense is that already it should act without fixed points. So, what is the way out, the way out is we slightly relax this proper this definition of properly discontinuous action and we give a slightly more general definition that is good enough to get give us what we want. So, let me propose that definition ok.

So, let me write that out. So, you see I am proposing this definition only for I am going to propose this definition only for let us say sub groups of Mobius transformations acting on some open sub set of the complex plane ok.

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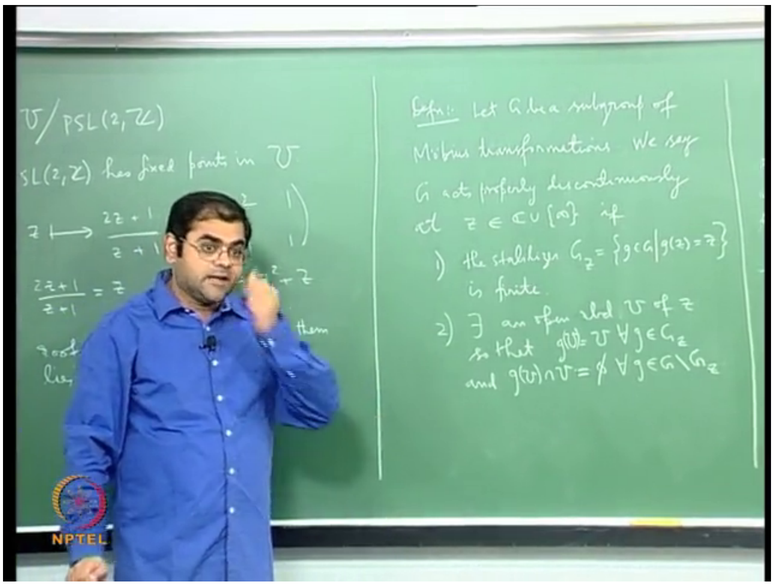

So, definition let G be a sub group of Mobius transformations. We say G acts properly discontinuously at a point Z in C union infinity. So, you see when you are looking at Mobius transformation you should also include the point at infinity, if you do not include the point at infinity then you are missing something because the Mobius transformation can easily take a finite point to a point at infinity. So, you have to look at the Riemann sphere actually. So, we say G acts properly discontinuous at a point if two conditions are required, in the first condition is see compared with our earlier definition in the earlier definition the there were no fixed points and there were no fixed points means that stabilizers are all trivial, because after all stabilizer of a point is all those elements of G which fix that point. So, if a group is acting without fixed points then the stabilizer are all trivial. So, we will relax that condition here and say stabilizers need not be trivial, but they have to be finite ok.

So, the first condition is the stabilizer these stabilizers G sub Z the set of all g belonging to G such that g of z is finite. So, this is the relaxation that you make instead of trivial stabilizer which is the case of a fixed point free action you make the stabilizer trivial. Then the second condition is with respect to that neighborhood that gave the proper discontinuous action what was the earlier situation the earlier situation was that apart from the identity element every other element of G has to completely move this neighborhood. Now what we do is we say that there is a neighborhood of the point which is completely fixed by every element in stabilizer and completely displaced by every element outside the stabilizer. So, there exists an open sub set open neighborhood U of Z. So, that you see g of U equal to U for all G in the stabilizer, and g of U intersection U is empty for all G outside the stabilizer.

So, there is one open neighborhood of the point which on which this finite group namely the stabilizer is acting and if you take any element of the group outside the stabilizer that is going to push it away. Now the beautiful thing is that this if you take the points the set of all points. Of course some groups may not have even a single point where it may they may act properly discontinuously in the relax sense, but it so happens that it happens for our P S L 2.

So, the general theory is now given a group of Mobius transformations, which acts properly discontinuously in this sense you can still go modulo that group and get something which is a more general case of covering it is called a ramified covering, you get what is called the ramified map of Riemann surfaces. So, what I wanted to tell you is that this is relaxation of this I mean of the earlier definition of properly discontinuous action and it is very nice I mean weakening of that definition, but strong enough to give the quotient of u mod P S L 2 Z.

And so this will lead us to trying to understand various types of groups it is; namely discrete sub groups Mobius transformation, then Kleinian groups, Fuchsian groups. We are going to look at all that because these are the general kinds of groups which we would like to modulo and try to get Riemann surfaces. So, we will do that in the forth coming lectures.