An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1 -dimensional Tori and Elliptic Curves
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Lecture - 24
Orbits of the Integral Unimodular Group in the Upper Half-Plane

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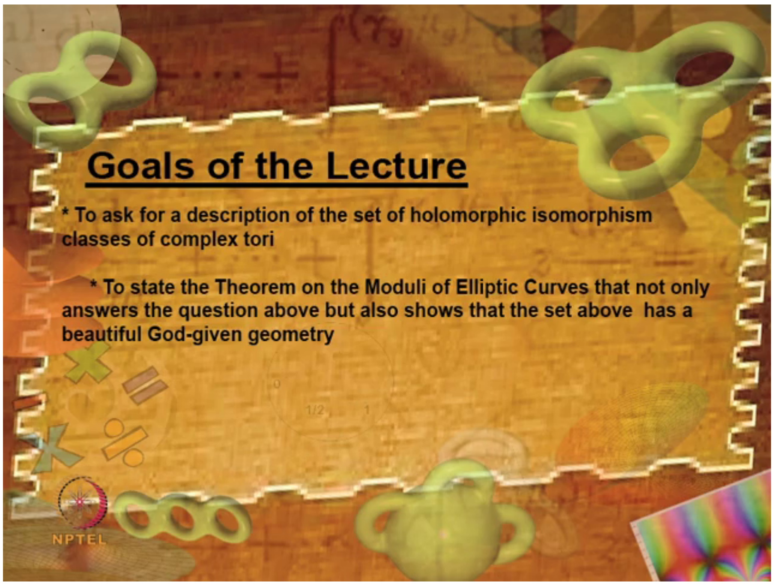

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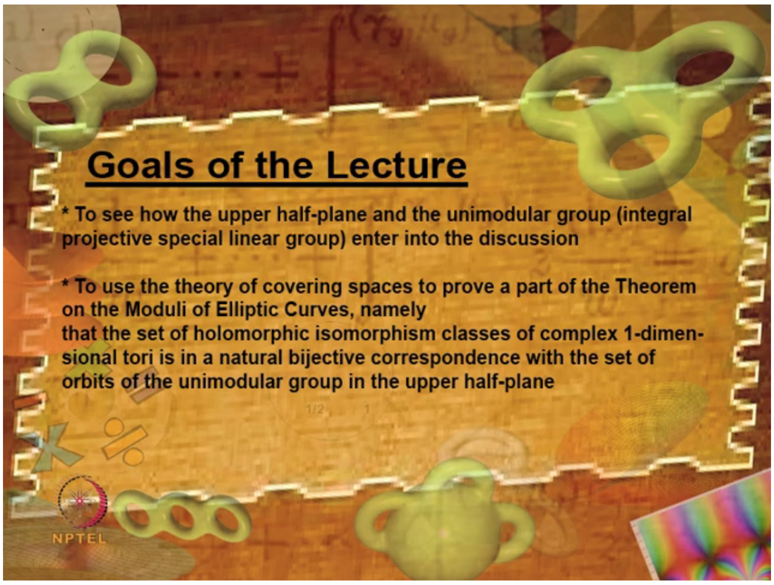

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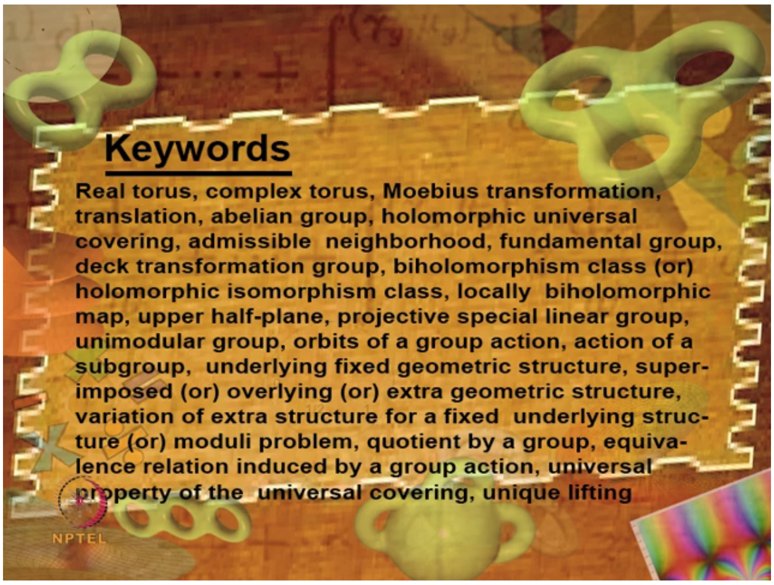

So, what I am trying planning to do in the next few lectures is to study the case of complex Tori. So, let me recall what a complex torus is.

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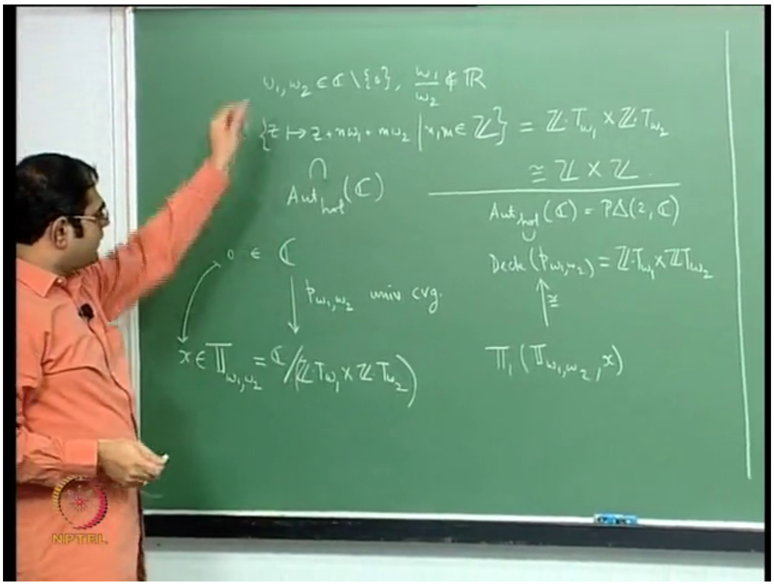

So, if you remember we choose two complex numbers omega 1, omega 2 which are nonzero. So, we fix this and assume that the ratio is not a real number which means there has their linear independent linearly independent over R. And then we look at the sub group of Mobius transformations which correspond to translation integer translations by integer multiples of omega 1 and omega 2. So, you look at z going to n times, so z going to z plus n times omega 1 plus m times omega 2 you look at all these translations by integer multiples of omega 1 and omega 2 integer linear combinations of omega 1 and omega 2, where n and m are integers.

And this of course, these are Mobius transformations. And you see being translations the only fixed point that they have is the point at infinity. So, these are these form a sub group of the holomorphic automorphisms of the complex plane. So, the point at infinity is mapped to the point at infinity the rest of it is going to be the complex plane, the complex plane is any finite complex number is mapped to a finite complex number. So, and you see this group this subgroup of course, is Abelian and it can be identified with z dot translation by omega 1. So, in fact, I can write it like this cross z dot translation by omega 2 right which is which is eventually which is actually isomorphic z cross z.

And what you do is you take the complex plane alright and then you go modulo this subgroup. So, you go c modulo this subgroup z translation by omega 1 cross z dot translation by omega 2. And this gives you the complex structure a Riemann surface structure on the torus which we call as let me put T sub omega 1 comma omega 2. So, these Riemann surface structure on this which is topologically a torus a real torus. And this map is a covering map which is universal covering, because C is simply connected. And this covering map if it call it as p omega 1 sub p sub omega 1 comma omega 2 then this a universal covering. And because of this the fundamental group of this identified with the deck transformation group of this cover and the deck transformation group is precisely this.

So, pi 1, so if I choose for example, point let say zero, and suppose the point zero goes to a point x here, then I have the identification fundamental group of this torus T sub omega 1 comma omega 2 based at x is identified with the deck transformation group of this covering, which is p sub omega 1 comma omega 2. And this deck transformation group is precisely is precisely this group of translation z dot translation by omega 1 cross z dot translations by omega 2 which is exactly the same thing.

So, in fact, I should not even write isomorphic in fact, it is actually equal to. And of course, this deck transformation group which is sitting inside the group of the holomorphic automorphisms of the universal covering space, which is p delta to C. If you write it out as matrices these are all elements of PSL to c with which are of upper triangular form, fine. So, the question is we want to look at various possible Riemann surfaces that you can get in this way which to begin with, you would get by varying omega 1 and omega 2.

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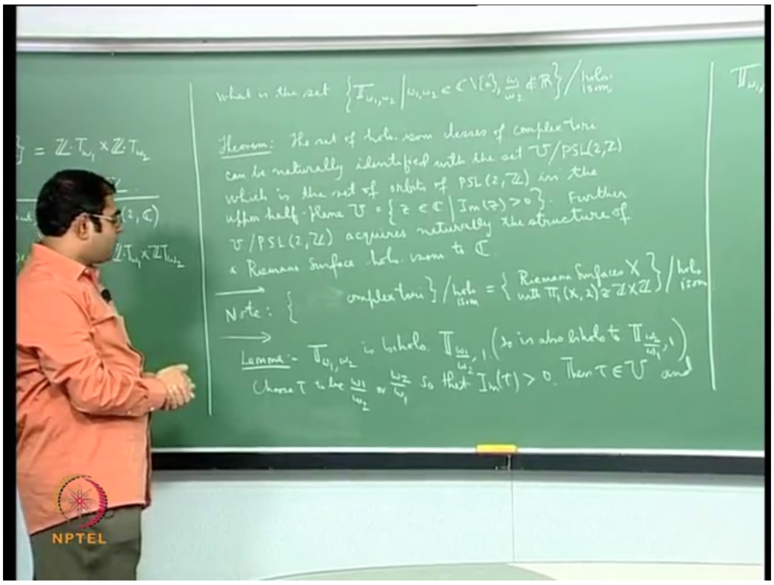

So, the question is what is the set of all Tori complex Tori omega 1 and omega 2, where omega 1 omega 2 are complex numbers nonzero and with non real ratio, omega 1 by omega 2 not a not a real number. So, this a set of possible complex Tori and then you want to go you go modulo holomorphic isomorphism so that means, I am looking at holomorphic isomorphism classes of complex Tori and I want to know how many of them are there namely I want to know what is this set.

Now, well of course, the answer to this question is the following theorem, which I had stated earlier. So, let me recall this the set of a holomorphic isomorph ismorph holomorphic isomorphism classes of complex Tori is can be naturally identified with the set u mod PSL 2, z which is the set of orbits of PSL 2, z in u in the upper half plane U. This is set of all z in C such that imagine it what have set is greater than 0, so that set is naturally identified viable with u mod PSL 2, z. And when you have a set modulo group what one means is of course, the orbits of the group on that side for the group acts on that set.

So, I want to tell you one thing namely recall that the holomorphic automorphisms of u are elements of PSL 2, r. And if and PSL 2, z is a sub group of PSL 2 r, and therefore PSL 2 z also acts on U. If a group acts on set in a subgroup also acts on the set, and you can also talk about orbits for that subgroup and that is exactly what you are doing here. And the fact is that if it is the orbits, this state is exactly the set of complex Tori up to holomorphic isomorphism this, the first statement.

Then there is more interesting statement is that this set U mod PSL 2, z acquires naturally the structure of a Riemann surface, and that Riemann surface happens to be isomorphic that is holomorphically isomorphic to the complex plane. Further U mod PSL 2, z is acquires naturally the structure of Riemann surface of holomorphically isomorphic to the complex plane. So, the up short of theorem is that this set is actually a Riemann surface and what is that Riemann surface is just the complex plane. So, it is amazing that this set has Riemann surface structure and that the Riemann surface is the complex plane. So, this is what I like to prove set how to prove in the next few lectures.

So, there are two tasks that we have to do. The first thing is how to identify this set with this set of orbits of PSL 2, z on U that is one-step. The next step is to explain how u mod PSL 2 z becomes Riemann surface, and how that Riemann surface is biholomorphic that is holomorphically isomorphic with the complex plane, so this is what we are going to do. So, I will begin with trying to show this set is bijective with two mod PSL 2, z. So, the first thing I want to state is that, so just for the sake of novelty note that that set of complex Tori modulo holomorphic isomorphism is also the same as well these are going to be notice let us recall theorem. If x is a Riemann surface with fundamental group isomorphic to z cross z, then x has to be complex to us. So, there is a theorem that we have proved.

So, I can just write this also as a set of I mean well I should not write set of and also put a bracket, but let me not be too pedantic about this. So, let me just say, so I hope it is all right. So, let me clear, let me rub this. So, here I can also write as Riemann surfaces x with pi 1 of x comma some base point, base point does not matter isomorphic to z cross z modulo holomorphic isomorphism. One can also state like this. You are looking at all those Riemann surfaces with fundamental group isomorphic to z cross z, and your studying them up to holomorphic isomorphism. In this sense what you are doing is well you are looking at various possible Riemann surface structures that you can put on a real torus, the underlying structure is topological real torus. And then you are looking at various Riemann surface structures that you can put on this and trying to look at how many non isomorphic structures you get.

So, there is an underlying structure which is the structure of the real torus. And there is a super imposed structure which is that of a Riemann surface. And the question is how many extra structures can you give for the same underlying structure. So, this is exactly what is called a moduli problem in a simplest form. You have a fixed structure on an object and then you know that you can put an extra structure on it, the question is how many extra structures on isomorphic extra structure that we can put on that, and if you look at the set of all those structures that have a geometry that is a question of modulo. And we get a beautiful theorem in the case of complex Tori. So, this is the first case where you get a beautiful theorem on moduli. So, this toned was just for just fun.

Now, well now I am going to do make a small, so the first thing is where does the upper half plane come into the picture. So, you can ask this question where does the upper half plane come into the picture it comes because of the following things because you can given any torus like that you can normalize it. So, here is a lemma, the lemma is T you take this torus T omega 1 comma omega 2, this is biholomorphic that is holomorphically isomorphic to the torus omega 1 by omega 2 comma 1, so which means I literally have these two complex numbers I divide throughout by omega 2. So, the second one becomes one the first one becomes omega 1 by omega 2. And in the same way is also biholomorphic to well the torus T of well omega 2 by omega comma 1 of course, I could have divided by omega 1.

So, now what I wanted to tell you is that, so I want to tell you what this lemma means with respect to U, you see omega 1 by omega 2 is not real is an non real ratio. So, it is a complex number it has an imaginary part. And then if you take the imaginary part, then either omega 1 by omega 2 or omega 2 by omega 1 has an imaginary part which is positive. And then you choose the one that has imaginary part positive and call that as tau and then that tau is an element of U. So, what I am trying to say is that choose tau to be well omega 1 omega 2 or omega 2 by omega 1, so that real the imaginary part of tau is positive. You can do that, and in one of them has to have if one of them has a imaginary part negative, then the other one will have imaginary part positive.

So, what is going to happen you choose the one it has imaginary part positive. Then you have tau in the upper half plane and T omega 1 comma omega 2 is biholomorphic to may does not enough space. Here let me write it in the next board then tau belongs to u.

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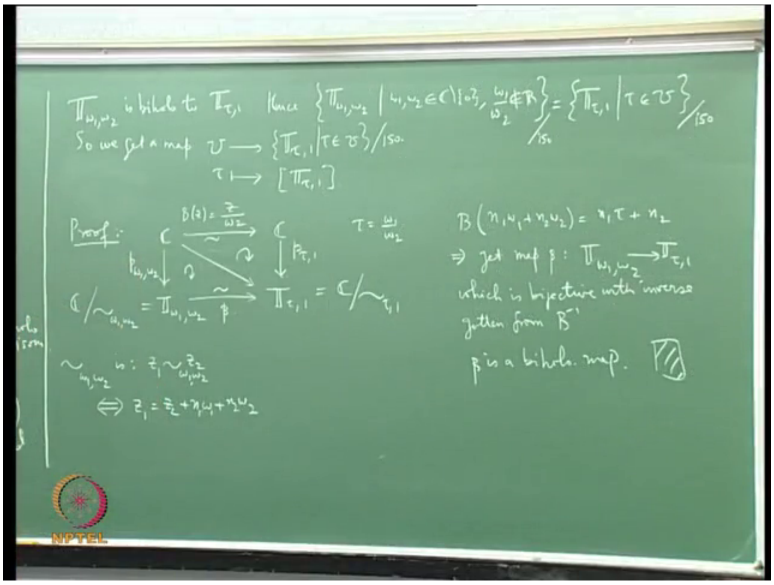

And the torus T omega 1 T omega 2 is biholomorphic to well T tau comma 1. So, the moral of the story is that instead of looking at all possible Tori like this, it is enough to normalize these two complex numbers in such a way that one of them is in element of the upper half plane and the other one is 1. So, what this lemma is says is that it is enough to look at T tau comma 1 with tau in u that is all, with tau in U. So, real part of tau is greater than 1.

So, what is the up short let me write that down hence set of T omega 1 comma omega 2, well omega 1 comma omega 2 in nonzero complex numbers omega 1 by omega 2 not real is exactly the same as the set of T tau comma 1 that tau is in U. So, so it is enough to look at only complex Tori of this form up to isomorphism. So, in fact maybe I should not I should not write like this I should write mod isomorphism, so mod isomorphism is equal to this mod isomorphism in fact, should write mod isomorphism not exactly set theoretically. So, instead of looking at isomorphism classes here I can also look at isomorphism classes here.

Now, what is the proof for this lemma the proof of this lemma, so that is where you comes into the picture that is where you comes into the picture. So, in fact, what is happening is that you, so we have, so let me write that down. So, we have actually a map. So, we get a map from U to the set of all Riemann surfaces, I mean set of all complex Tori mod isomorphism namely you just sent out to the isomorphism class of T tau comma 1. So, you have map like this. And now you know what it is that you are trying to prove, we are just trying to prove that this map factors to U to the quotient U mod PSL to z which is the orbits of PSL 2 z in U. And when it factors it gives raise to bijective map with this that will tell you that the set of isomorphism classes of complex Tori is naturally identifiable with the orbits of PSL 2 z on the upper half plane.

Well, so let us give the proof of the lemma; it is pretty easy to prove. So, what we have is well. So, let me write out the universal coverings for these two. So, I have C to well I have this projection omega 1 comma omega 2, then here I have the torus complex torus defined by omega 1 and omega 2. And I have on the other hand C to that is this projection tau comma 1, and I have the complex torus tau well tau comma 1, where tau is where let me take tau to be omega 1 by omega 2, if you want to. Because I could have also equally taken tau is equal to omega 2 by omega 1.

Now, what you do is define this map B of z is well z by omega 2. Look at this map phi of z is equal to z by omega 2 now that map I can divide by omega 2 because omega 2 is not zero. And then it is going to take the complex plane out of the complex plane, it is a Mobius transformation and it has, so it is a holomorphic automorphism of C it is a holomorphic automorphism of C. And let us see what it does see notice that this just c mod an equivalence relation which is this the equivalence relation is, so this there is an equivalence relation let me put it equivalence sub omega 1 comma omega 2. And what is this equivalence sub omega 1 comma omega 2 is you see z 1 is equivalent under this under this relation to z to if and only if z 1 is z 2 plus an integer linear combination of omega 1 and omega 2 let me say n 1 omega 1 plus n 2 omega 2. And this is exactly the equivalence. I mean this another way of saying that you are going modulo translations by integer multiples of omega 1 and omega 2.

And well and you have a similar statement here this is the complex plane modulo the equivalence relation given by tau comma 1. And well the definition is similar to this. And now what I want you to understand is that what does B; do to a point like this. If you remember these were points of these were the points were grid of parallelograms. And we were actually going modulo that grid and see what B does to a point like this B takes n 1 omega 1 plus n 2 omega 2 to well I do be is supposed to the division by omega 2. So, what I am going to get, I will get n 1 tau plus n 2, I will get n 1 tau plus n 2.

But n 1 tau plus n 2 is a point of this grid. The grid here is the grid which is given by tau and one. You form a parallelogram with just one and tau, and then repeat this parallelogram you get a grid of parallelograms, and the end points of the grids vertices of the grid are precisely these elements of this form. So, what B does is it takes this grid to that grid. And because of the fact that these are just because of the fact that these are equivalence relations modulo going modulo those grids, you are going to get a map like this. I am going to get a beta; I am going to get a beta such that this diagram commutes. So, this implies that at a set theoretically set theoretically I am going to get a map beta get map beta from this torus with torus.

Now, the point I want to make is that of course, this map is bijective map, because I have B inverse, these isomorphism B inverse will be multiplication by omega 2. So, B inverse will also map a point like this to a point like this that will also take this grid here to the grid here. So, B inverse is going to induce an inverse beta. So, beta is actually a bijective map, so which is bijective with inverse gotten from B inverse in exactly the way we got the beta from B. So, I get a bijective. Now, I want to say that this bijective map is actually biholomorphic map. So, why is it biholomorphic, it is biholomorphic for the following reason.

So, let me break this diagram into two commutative diagrams. So, this B followed by beta 1, this is holomorphic, and this is holomorphic. Mind you the holomorphic structure on this was inherited from the holomorphic structure above, and once you fix this holomorphic structure this covering projection becomes a holomorphic map, it is a covering in the holomorphic sense. Therefore, this is holomorphic, this is holomorphic therefore, this is holomorphic and this map is a covering map. So, it is locally invertible it is locally holomorphic.

Therefore, this followed by this local, but then again this is also a composition of holomorphic maps. And therefore, this is holomorphic. So, the moral of the story is that the map beta is holomorphic. So, it is an injective holomorphic map. So, it is a biholomorphic map. So, beta is a biholomorphic. And this literally tells you that this torus and this torus are holomorphically isomorphic. So, it is very simple lemma. So, the advantage as I told you this lemma is that it is enough to look at only Tori which are given by points on the upper half plane.

Now, let me go ahead and try to say that we get an identification of these isomorphism classes of this set of isomorphism classes of these Tori with u mod PSL 2 c. So, well, so suppose, so let me start the argument like this, because I think it will be very ideal for the exposition yes correct.

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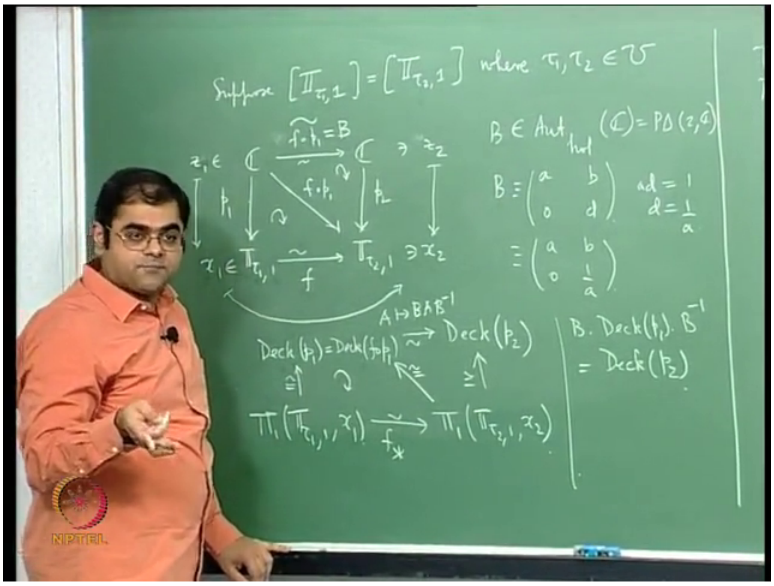

So, suppose the isomorphism class of T tau 1 comma 1 is equal to the isomorphism class of T tau 2 comma 1, where tau 1 and tau 2 are the (Refer Time: 31:06), suppose they are isomorphic. So, this means that the torus defined by tau 1 isomorphic holomorphically isomorphic torus it defined T tau 2. Now, again let us draw the covering maps. So, I have c. So, I have this p one and this T tau 1 comma 1 then I have again C, this pi 2 and I will have T tau 2 comma 1. And since it is given that these two are holomorphically isomorphic let me choose an isomorphism, let me call it something let say f. So, I am given the holomorphic isomorphism f between these two torus. And what I am going to do is using this and covering space theory I am going to cook up an element of PSL 2 z that moves tau 1 to tau 2 that is what I am going to do.

So, how I am going to do this? So, you see because of this map, so let us look at this whole diagram. The first thing is as we have done in the case of as we have done earlier see if I take this composition writing this composition, this is a holomorphic isomorphism this, the covering. Therefore this continuous to be a covering. So, this is just p 1 followed by f that is what this and this also a covering. And you know by the universal property, universal lifting property lifting property is of coverings this lifts to a map here.

So, for that I will have to do some work I will have let me choose 0 here and let us assume that 0 goes to let us say a point x 1 here. And let me assume that f of x 1 go f takes x 1 to x 2 and let me fix a point, let me call this as well I do not even need to put 0 may be I put z 1. Let me call this as z 2, fix a point z 2, which goes to x 2 under. So, p 1 takes z 1 to x 1 p 2 takes z 2 to x 2 f takes x 1 to x 2. And I can lift this f circle p 1 to a map f circle p 1 tilde which I would like to call as let me call this as B such that this diagram commutes. And well this map B is going to be holomorphic isomorphism because I can construct an inverse, I can construct the inverse simply by taking f inverse just as I got B from f, I will get B inverse by looking at f inverse because f is an isomorphism.

So, well B is an element of it is a holomorphic automorphism of C, so element of an p delta 2, C. So, B can be identified with you can give it a matrix representative in PSL 2 z namely a b 0 d and it is in PSL. So, a d is 1, so d is 1 by a. So, B will look like. So, this will be just look like a b 0 1 by a. And of course, so this means that B is a Mobius transformation B of z is equal to a z plus b by 0 z plus T, which is how all automorphisms of holomorphic automorphisms of p look like.

Now, one is to look at what is happening to the deck transformation group because of this. So, this is a map this a whole diagram of spaces. You know that forming the fundamental group is factorial. So, I am going to get a corresponding diagram for the fundamental groups. So, let me write that down. So, what I am going to get is I will have here the fundamental group, first fundamental group of this torus based at x 1 that is going to identified with the deck transformation group of this covering, which is deck of p 1. And this f is going to give me an isomorphism f lower star which is you know take a loop at x 1, you can produce a loop at x 2 which is just well you take just take the image of this loop there, and do this for homo topic classes. So, this f lower star is what you get naturally because the formation of the fundamental group is factorial.

And here what I will get here is fundamental group of this the other torus T tau 2 comma 1 based at x 2, and that is well identified with the deck transformation group of p 2. And you have also one more, there is also a map like this. So, this also a covering; because of this covering I will get an identification of this the fundamental group of this namely this with the deck transformation group of this cover, but then followed by this is this. So, what will actually tell me is that this same as the deck transformation group of f circle p 1, these will be one and the same. This is for example, I prove this explicitly in the previous lecture.

So, this is another identification, this identification comes because of this covering; and because this diagram commutes this diagram also commutes and that is the reason why these are the same and what about this map. So, this map is just you know conjugation. Namely, if you give me deck transformation here how do I get a deck transformation there? What I do is I go like this apply the deck transformation and then come back going like this applying B inverse then applying the deck transformation here and then applying B. So, it just the map that sends A to B A B inverse is just conjugation by the and this is an isomorphism because this map is an isomorphism this a holomorphic isomorphism. So, well, so the up short of all this the following I have d dot deck p 1 B inverse is equal to deck p 2. Namely, the deck transformation group of p 1 and p 2 are deck, they conjugates in the group of holomorphic automorphisms of C. So, I get this.

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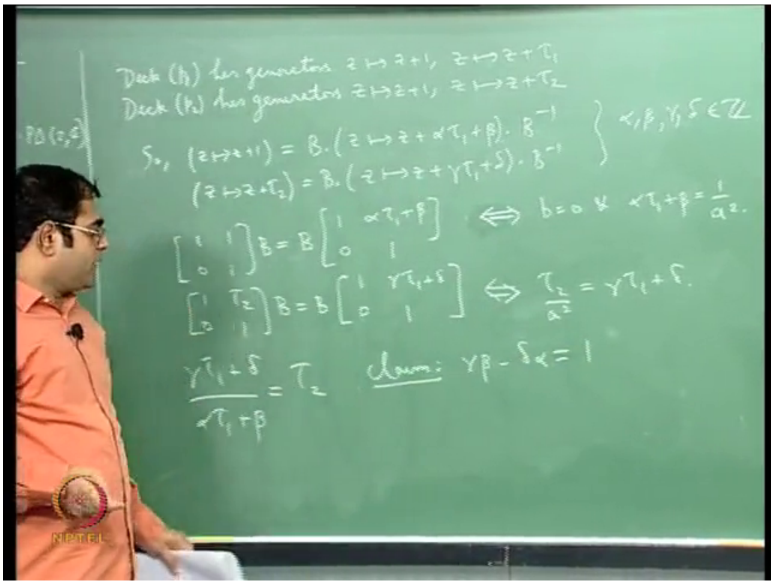

Now, let us go and look at what the generators of this of these deck transformation groups we chosen as and will get a little bit more we can exactly little bit more of information. So, well, so we have see deck p 1 the deck transformation group of p 1 is has generators z going to z plus 1 z going to z plus tau 1. So, these are the generators of deck transformation group, after all the deck transformation group is precisely the sub group of holomorphic automorphisms that you have to go mod to get this torus and sub group of holomorphic automorphisms here are translations by integer multiples of tau 1 and 1. So, you get this. And similarly deck p 2 has generators z going to z plus 1, z going to z plus tau 2. Notice that these two are they are isomorphic to the corresponding fundamental groups in the fundamental groups such as z cross z, therefore, there are two generators.

Now, you see, so if you look at this, what this will tell you is that if I take z going to z plus 1, then it is a conjugate of an element here and similarly z going to z plus tau 2 is a conjugate by B of an element here. So, let me write down what I worked out. So, well, so you can write z going to z plus 1 is equal to B z going to z plus alpha tau 1 plus beta dot B inverse and z going to z plus tau 2 is B dot conjugation by B of z going to z plus gamma tau 1 plus delta dot B inverse. Where alpha, beta, gamma, delta are integers. After all you see if you take z going to z plus 1 it is conjugate by B if an element here, but how does an element here look like it is translation by an integer multiple of 1 and tau 1. So, it will look like this. Similarly, z going to z plus tau 2 is going to look like conjugate by B of an element here, which I have taken it in this form that is why you get this four integers, you get these four integers. And you can guess that they are going to give you the element of PSL 2 z, so that is a calculation that one has to do.

So, let me write it down. So, what happens is if you write it down this is what you get. So, you see suppose I write it in matrix form this 1 1 0 1 d is equal to B 1 0 1 alpha tau 1 is beta I mean this I just this written in matrix form I have just pushed this B inverse to B on the right; I have multiplied both sides of the equation on the right by B. And if I write it in the matrix forms what I get. Similarly, if I do it in the second equation what I will get is I will get one tau 2 0 1, B is equal B well is equal to B times the low one which is 1 0 1 gamma tau 1 plus delta, I get this.

If you write out, if you make the comparisons and write out, you will see that this is true these two are true. So, this is true if and only if you will see that b is equal to 0 and you get alpha tau 1 plus beta is 1 by a square, where you know this a was this a that we chose for the matrix representative of B and PSL 2, c. And here you will get these two are equal I mean this equivalent to well again of course, B is 0 and you will get tau 2 by a square is equal to gamma tau 1 plus delta, this is what you get.

Now, you can see immediately that you know if I write if I eliminate a square, what I am going to get is I am going to get gamma tau 1 plus delta by alpha tau 1 plus beta is equal to tau 2. So, what we have gotten is an element of PSL 2, z which is moving tau 1 to tau 2. Now, there is a point here there is a check that has to be done to say that gamma beta, gamma minus alpha delta is actually equal to 1. One has to for example, even verify that this really a Mobius transformation one has to really verify that gamma beta minus alpha delta is equal to one is for example, if not equal to 0.

So, how does one do that? So, the claim is gamma beta minus delta alpha is actually equal to 1. So, for this it is enough to prove that gamma beta minus delta alpha is nonzero. See gamma beta minus delta beta is not zero, it is an integer and it is it has to be invertible, because it is an determinant of an integer matrix, it has to be an invertible integer. So, if you prove gamma beta minus delta alpha is not zero, then you will get gamma beta minus delta alpha equal to plus or minus 1, then I will have to eliminate the case that it is minus 1. And eliminating the case that is minus 1 is precisely where I am going to use the tau 1 and tau are have real part I have I have imaginary part greater than zero which is very very direct calculation.

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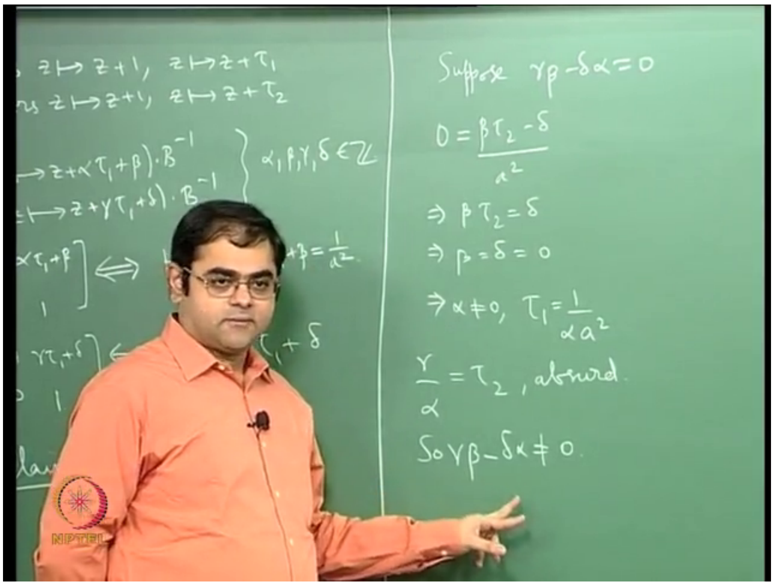

So, let me explain. So, let me write down that calculation. So, suppose it is 0, I mean I am trying to give a very elementary argument, suppose gamma beta minus delta alpha is 0. Then what you do is you use these two equations, you use these two equations and try to use the fact that so you know I multiply this equation by delta. So, I can get delta alpha. Then I multiply this equation by beta, so that I can get a beta gamma and then if I subtract alright then this delta alpha tau 1 minus is beta gamma tau 1 is going to go away.

And if I do that what I am left out with is zero equal to beta tau 2 minus delta by a square this is what I will get. Multiply this equation by delta, multiply this equation by beta and then subtract you will get this. And you have use gamma beta minus delta alpha is zero you will get this. But this will tell you that you know beta tau 2 is delta which is impossible, because you see beta is an integer, delta is an integer, tau is an complex number. It will tell you that tau 2 is equal to for example, if beta is not zero it will tell you that tau 2 is a rational number, a real number for that matter it is which it is not because tau 2 is imaginary. So, this will imply the only way is beta is equal to delta is 0, there is no other way.

But you see if beta is equal to delta is 0, see for example, you know if beta is equal to 0 then alpha is not, if beta is 0, alpha tau 1 is 1 by a square that is not 0. So, alpha is not 0. And what you will get if I put it here I will get gamma. So, I will get alpha is not 0 and tau 1 is equal to 1 by alpha a square. And if I put it back into this, and use the fact delta is 0, I will end up with getting gamma by alpha is equal to tau 2. So, I just delta is 0, I cross multiply these a square and a square tau 1 I write it as 1 by alpha then I will get gamma by alpha is equal to tau 2 which is again non zero. Because you see now tau 2 becomes the rational number gamma and alpha are integers, so that is not possible absurd. So, it is a very elementary argument to show that alpha gamma beta minus delta alpha is nonzero. So, gamma beta minus delta alpha is not equal to 0.

So, now we need to see why gamma beta minus delta alpha is actually equal to 1. So, for that what one needs to do is to look again to look again at this diagram here, one has to look at this diagram here and look at it carefully. The first thing that we should remember is that both deck transformation groups you see they are you know they can be identified as z modules with complex numbers. So, what you can do? So, one can write down the following. So, let me write down that diagram again.

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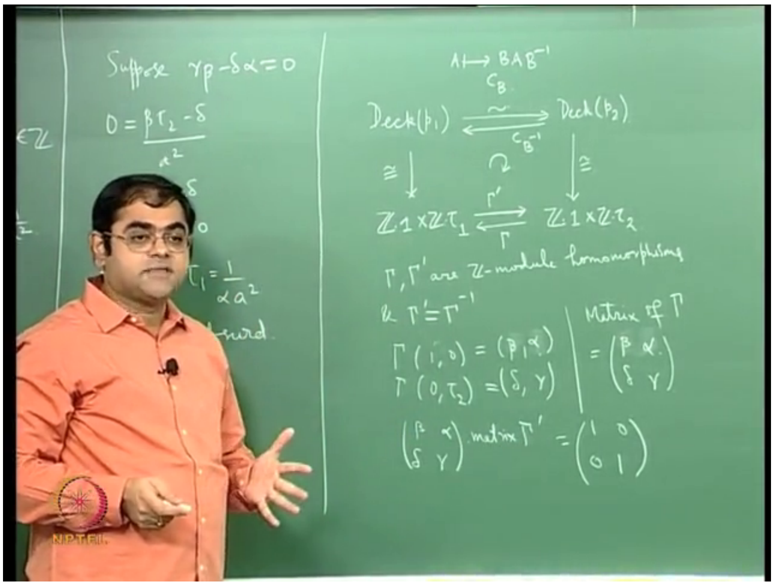

So, I have deck p 1 right point here then I have this isomorphism, which is conjugation by B this just the map a going to B A B inverse. And this is an isomorphism of that of the deck transformation group of the covering given by p 1 with the deck transformation group of the covering given by p 2. And well you see this deck transformation group can be identified you see with this is an isomorphism with z dot. So, let me follow the follow what I have written down z dot one are cross z dot tau 1.

So, what I am doing here is an element here is translation by an integer multiple of one and an integer multiple of tau 1. So, it is given by two a pair of integers and I am just identifying that with these two integers here. So, there is an isomorphism like this that makes a so this isomorphism actually shows that this a z module this is an isomorphism with z modules. So, similarly, I have an isomorphism here, this isomorphism will be in z it go to z dot 1 cross z dot tau 2. Of course, this because of this you see this deck p 1 has generators z going to z plus 1 and z going to z plus tau 1 and deck p 2 has generators z going to z plus 1 and z going to z plus tau it is only. And you see these are generators as z modules. So, you have this identification.

Now, if you look at one would like to look at this map which is given by the commutativity of this diagram. So, well notice that this map is conjugation by B it has an inverse the inverse map is just conjugation by B inverse, obviously and this conjugation by B induces let me call this map, so mind you this map is also going to be an isomorphism it will be an isomorphism of z modules. This is also an isomorphism. And I will get.

So, this let me call this as gamma prime and then I will have map like this. So, the conjugation by B inverse is going to induce B like this, namely this map is this followed by this followed by this that is this map; and this map is this followed by this followed by this. So, I have these gamma and I have gamma prime, and it is obvious that you know gamma and gamma prime are z module homomorphisms and they are inverses of each other gamma, gamma prime are z module homomorphisms; and of course, gamma prime is gamma inverse by the very way it is defined.

Now, the important thing is what is the matrix for gamma. See gamma is the map from two copies of z 2 two copies of z. So, it is given by a 2 by 2 matrix with integer in z. And what is that map, and that map is see claim is this gamma, so let us calculate. So, take the element one here take the element one here, see this I need to know what where gamma of one goes to, so take the element one here this element one is a identified with the transformation z going to z plus 1. And that transformation will go to B inverse dot z going to z plus 1 dot B, and then I am going to identify it with this. But what is go back to this equation here B inverse dot z going to z plus 1 dot B is actually z going to z plus alpha tau 1 plus beta.

So, when I come here, I will get the map z going to z plus alpha tau 1 plus beta. And if I bring it down here I will get the elements alpha times one plus beta times tau 1, so I basically get alpha comma beta. So, gamma of 1, if you want what I mean by that is gamma of one comma 0, you take one here you take 0 there alright. This is alpha comma beta and what is gamma of 0 comma tau 2, what is gamma of 0 comma tau 2. See take tau 2 here it will go to the deck transformation which is translation by tau 2, so it will go to z going to z plus tau 2 alright and here it will go to it is conjugate by B inverse. So, namely I will get B. So, now, go back to this diagram see B inverse z going to z plus tau 2 B is actually z going to z plus gamma tau 1 plus delta.

So, the image of tau 2 here will be the translation by gamma tau 1 plus delta. And you see if I take its image here I am going to end up with delta, so it is delta comma gamma is delta comma gamma, if I write it in this order gamma plus delta tau 1, that is what it is. So, you see what the this tells you that matrix of gamma is alpha beta delta gamma and I think I am not writing it correctly. So, it is alpha times wait a minute, alpha is coefficient tau 1, so beta alpha I should write it here also. It beta times one plus alpha times one, it is beta this the matrix of tau.

And you see for gamma prime also, similarly you will get an integer entries. And what will happen is you see the whole point is gamma and gamma prime are isomorphisms their z module isomorphisms. So, the fact is so, let me write this here beta alpha delta gamma times let me write matrix of gamma prime; if I compose, I should get the identity matrix. And this is I get this because these this the inverse of gamma prime gamma is the inverse of gamma prime that is right gamma prime inverse is gamma, gamma inverse is gamma prime.

So, I get this. And the matrix of gamma prime will also have integer entries I mean it is you have to write the same way as you wrote gamma. But in any case what will happen is see this is the equation that helps me because now if I take determinant if I take determinant and notice that all my calculations are happening in z, they are happening in z the integers.

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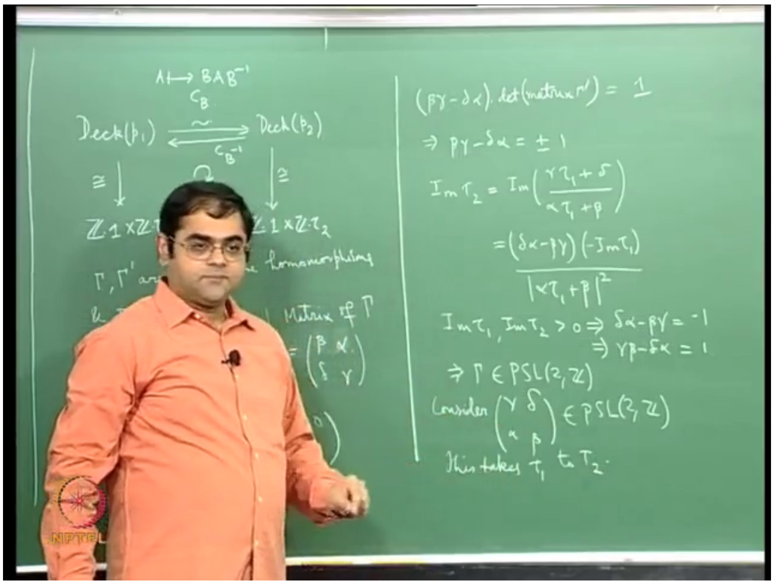

So, if a determinant I will get what I will get is a I will get beta gamma minus delta alpha times determinant of matrix of gamma prime is equal to 1. Now, you see I get a product of two integers equal to 1. So, the possibility is that either both are plus 1 or either or both are minus 1. So, this will tell you that beta gamma minus delta alpha is equal to plus or minus 1. Now, of course, one can rule out the case that beta gamma minus delta alpha is minus 1 because of the condition that tau 1 and tau 2 have imaginary part positive that is that is because of the following calculation.

So, let me write that down. So, here I am. So, if I calculate imaginary part of tau 2 it is by definition imaginary part of well tau 2 is now going to be I have a written it there, I have written it here, tau 2 is gamma tau 1 plus delta by alpha tau 1 plus beta. So, gamma tau 1 plus delta by alpha tau 1 plus beta, this is what it is. And you write the imaginary part of this as this minus it is conjugate divided by 2 y, you just write it out. And if you simplify it what you will end up with this you will get delta alpha minus beta gamma times minus of imaginary part of tau 1 divided by modulus of alpha tau 1 plus beta the whole square this is what you will get. If you make a very simple calculation, you will get this.

And you will see that you see the imaginary part of a tau 2 is positive, imaginary part of tau 1 is positive, and this quantity is also positive. Therefore you see this quantity has to be negative. So, this will tell you imaginary part of tau 1, imaginary part of tau 2 positive will tell you that delta alpha minus beta gamma has to be minus 1, so that will tell you that well as we wanted gamma beta minus delta alpha is equal to plus 1 is equal to 1. So, this confirms that you know this matrix, so this means that your matrix gamma is actually an element of PSL 2. So, gamma is an element of PSL 2, it is a element of PSL 2 and you can take the image defined take it is image in the quotient group PSL 2.

Well, let us consider this namely the following thing gamma delta alpha beta, this is an element of PSL 2 z, this is also an element of a PSL 2 z. Again you see this takes tau 1 to tau 2 that is what I want. So, well if I chosen these constants correctly, I would have directly got the alpha beta gamma delta, but any way it does not matter. So, the point is I got an element of PSL 2 which takes tau 1 to tau 2. So, the up short of the story is if the torus complex torus is defined by tau 1 in the upper plane is isomorphic to the complex torus defined by tau 2 in the upper half plane, then there is an element of PSL 2 z which takes tau 1 to tau 2.

Now, the point is that this whole argument can be reversed. Conversely, if I am given a PSL 2 element which takes tau 1 to tau 2, then one can work out and one can so in that case you know alpha beta gamma delta are given integers such that you know gamma beta minus alpha delta is equal to one and given this. So, in some sense, so I am given gamma; and from gamma I have to cook up, I have to cook up an isomorphism, I have to cook up an isomorphism between these two Tori. And how do I do this?

Well, I am given this gamma, what I can do is first I can first cook up, this I can cook up this B, which is given by the quotient. And how do I get these quotients, well you see I set if you go back here I set small b equal to 0, and then I set a to be as square root 1 by alpha tau 1 plus beta. So, I set a like that. And then I consider this transformation a 0 0 and instead of d I put one by a that will give me Mobius transformation B. And that Mobius transformation you can check will give me a map that will give me an isomorphism from C to C.

Now that isomorphism because of that isomorphism that isomorphism will actually go down. The reason why it will go down is because the lattice defined by 1 and tau 1 namely the z sub module of C defined by 1 end tau 1 will be mapped to the z sub module C defined by 1 end tau 2. That is because of the way that is because you are given that element of PSL 2, which does this. Therefore, this you will get this B, and it will go down to a map like this. You can cook up the B given alpha beta gamma delta and PSL 2 matrix, which takes tau 1 to tau 2 namely for which this holds I can write down this explicitly.

All I have to do is I have to set small b is equal to 0; I have to take instead of d of course, I have to put 1 by a. And for a, I have to take just square root of 1 by alpha tau 1 plus beta that is all I can that is all I have to do. And then I will get this B and you can check that this B takes any integer linear combination of tau 1 and 1 into an integer linear combination of tau 2 and 1 that is because of the way that is because of the unimodular transformation alpha beta gamma delta that is already given to us.

So, you will see that it will carry grid defined by 1 and tau 1 to the grid defined by 1 and tau 2. And therefore, you know it will induce the map like this and the d inverse will induce the map in the other direction. So, the map will be automatically bijective. The only question is why is it holomorphic isomorphism it is very simple because you see what will happen is well if I am given B then B for this of course, holomorphic it is a it is a Mobius transformation, then this also holomorphic. Therefore, this becomes holomorphic.

Now, this map is because this a covering this map is locally this followed by this because you see this map p 1 is a local homeomorphism it is a local biholomorphism. So, locally this map can be written as inverse of this followed by this by taking an admissible neighborhood at any point. So, this map becomes a composition of two holomorphic map. So, therefore, this becomes holomorphic in the similar way it is inverse also becomes holomorphic therefore, these two Tori becomes isomorphic biholomorphic. So, everything can be the whole argument can be reversed the whole argument can be reversed. So, the up short of the story is a following result.

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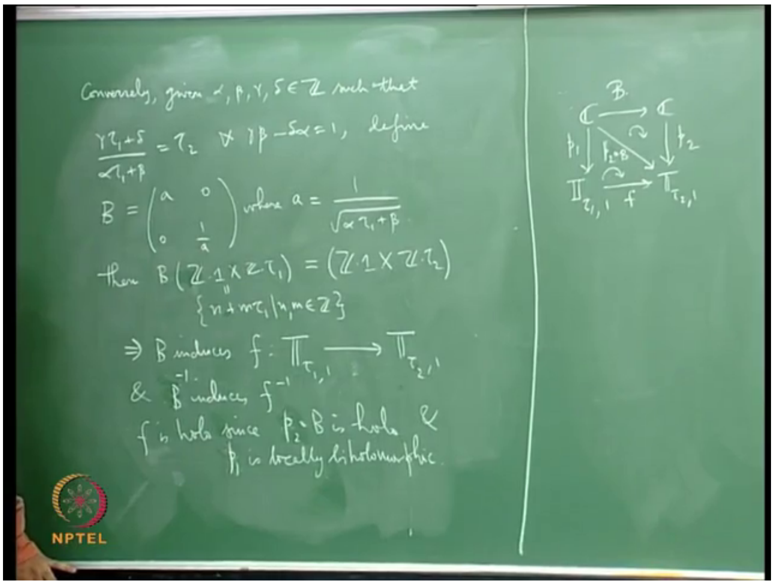

So, let me write that down. So, let me write this, just for the sake of completeness. Conversely given alpha, beta, gamma, delta integers such that gamma tau 1 plus delta by alpha tau 1 plus beta is equal to tau 2 and well gamma beta minus delta alpha is equal to 1. Define B by a 0 0 1 by a, where a is just given by this formula 1 by root of alpha tau 1 plus beta then B induces then B takes z dot 1 cross z dot tau 1, it takes this subset of complex numbers.

So, this is set of all complex numbers your integral multiples of 1 and tau 1. So, it is of the form n plus n plus m tau 1. So, this will go into z dot 1 cross z dot tau 2. So, this will imply B induces f from the torus defined by tau 1 to the torus defined by tau 2. And B inverse of course, B inverse induces f inverse. The only thing that is left is this see that f is holomorphic that is because f is holomorphic since. So, let me draw again draw a small diagram and rub this side.

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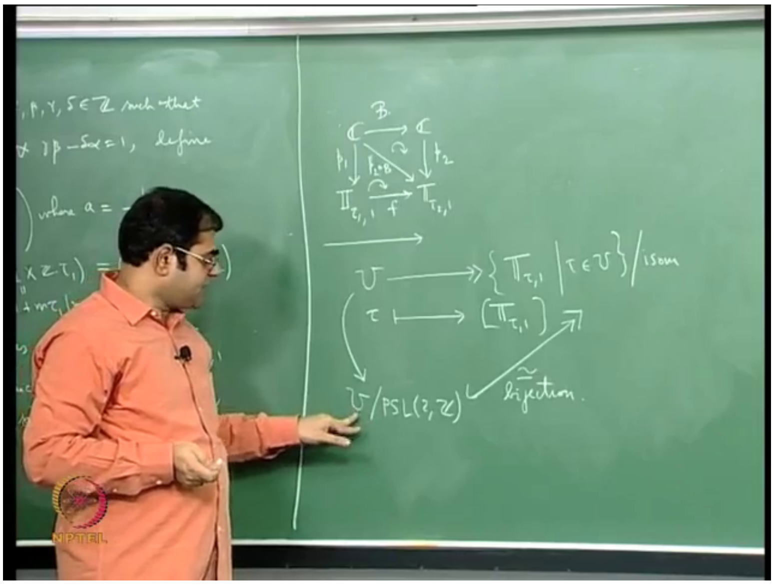

So, let me draw the diagram and then come back here. So, I have C to C, I have B and this p 1, this is a covering projection for torus defined by tau 1. And this p 2 the covering projection defined by torus defined by T 2. And this my f and this diagram commutes. And well this map is going to be the p 2, p 2 circle B that is what this map. So, it is this followed by this. And well of course, this diagram also commutes. So, what I am going to say is that f is holomorphic since B circle p 2 circle B is holomorphic, this is holomorphic, because it is a combination of holomorphic and because p 1 is locally biholomorphic and p 1 is locally biholomorphic. Therefore, f is an injective holomorphic map. So, it is a biholomorphic map. And therefore, we have proved that the Tori are isomorphic.

So, you see therefore, the moral of the story is that what I can do is from U to the set of all complex Tori isomorphic class of Tori which was of the form T tau comma 1, tau belonging to U mod isomorphism. I have this map tau being sent to T sub tau comma 1 isomorphism class biholomorphic class holomorphic isomorphic class. Then this map goes down to a map to the quotient U mod PSL 2 Z. So, I have the quotient U mod PSL 2 Z, this is the set of orbits of PSL 2 Z on the upper half plane. And well two points go to the same point here if and only if they are moved movable by an element of PSL 2 Z that is what we have proved. So, you get a map like this, you get a map like this.

Now, this map is of course, by definition subjective, therefore, this also subjective and it is also injective because for the simple reason that you know if you have tau 1 and tau 2 which give raise to isomorphic Tori then tau 1 and tau 2 have to be in the same orbit of PSL 2 z; so this also this as this injective as well as subjective, so this bijection. So, you see this completes the proof of this fact that the set of isomorphism classes holomorphic isomorphic classes of complex Tori is bijective to this set namely the sets of set of orbits of PSL 2 z in U. So, this is the first part of this story.

Now, what we need to do in the next what we will do in the next lecture is to show that this a Riemann surface, you have to show that this is Riemann surface. And that this map this bijective map, one wants to show that this Riemann surface and one also wants to show that this actually and Riemann surface that it is actually the complex numbers that is what one wants to say show, so that is the next part of the story.

So, we will continue in the next lecture.