An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1 -dimensional Tori and Elliptic Curves
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Lecture - 23
Classifying Annuli up to Holomorphic Isomorphism

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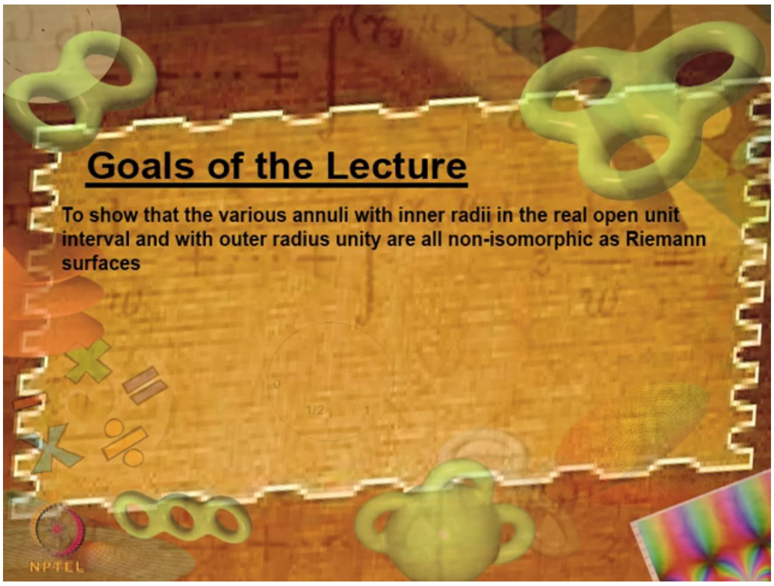

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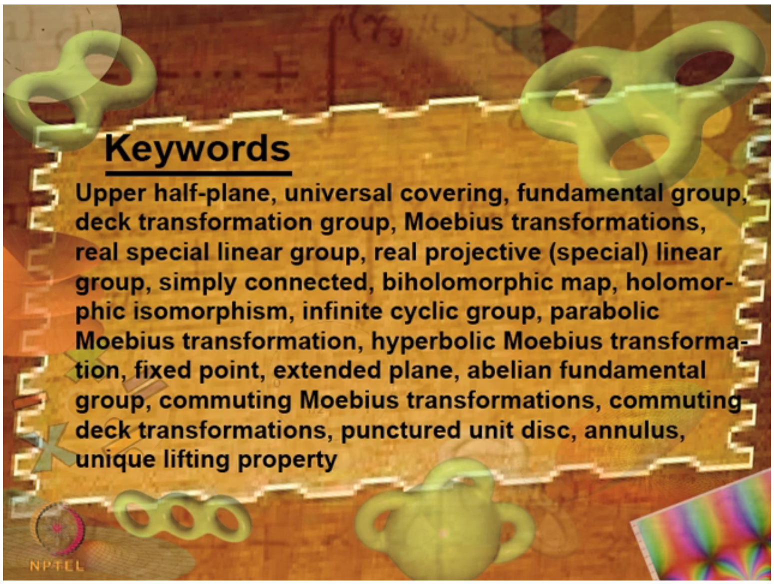

So, last time we were looking at the classification of Riemann surfaces which had the upper half plane as the universal covering plane. And, if you remember let me recall: we have the following cases.

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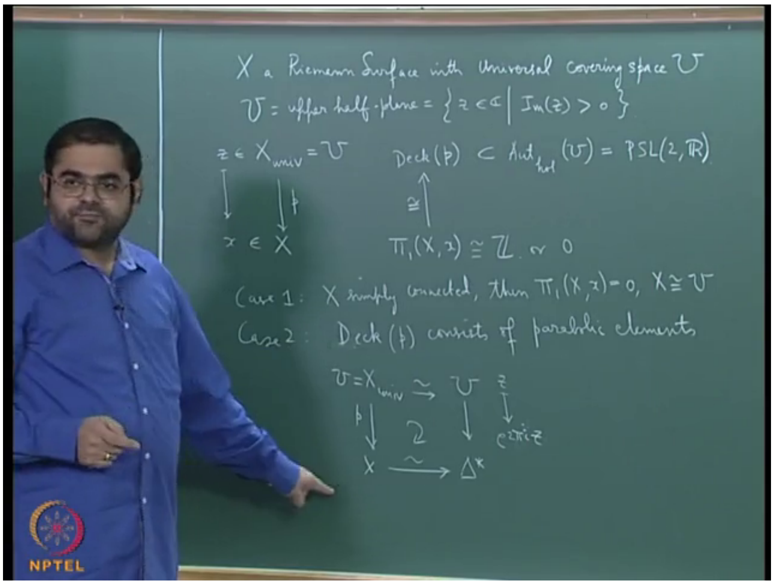

So, let me write them down X a Riemann surface with universal cover covering space U. So, U is the upper half plane which is the set of all complex numbers z such that imaginary part of z greater than 0. So, you have the picture, we have the following picture; this is X sub unit which is universal covering space this is U and here is the covering projection and here is X and as you know that if I dix a point z here which goes down to a point X here, then we have an identification of the fundamental group below with deck transformation group above.

So, you get this identification I 1 capital X comma small x; the fundamental group of capital X based at small x, you have an identification namely an isomorphism of groups with the deck transformation group of this covering. So, I will is a deck transformation group of this covering map. So, I will just put the covering map, I will just write as p for convenience I will not write the whole the source of the target and this is subgroup of the auto-morphic holomorphisms of the universal covering space U which as you know is identified with PSL 2 R. So, these are Moebius transformations of the form z going to a z plus B by c z plus c z plus d with a B c d real numbers real entries and a d minus B c equal to 1, right.

So, now there were 3 cases. So, of course, case 1 was the trivial case, X simply connected; if X is simply connected then the fundamental group is and X to X to X identity map itself is the universal covering therefore, by uniqueness of the universal covering X has to be isomorphic to U itself then pi 1 X comma X is 0 X is isomorphic u, this is this isomorphism is holomorphic; isomorphism is biholomorphic to you; case 2 was X. So, the we had 2 possible cases apart from case 1 that is we found that when that the fundamental if it is not trivial, then it is infant cyclic and there were 2 cases namely it consists can consisted of only parabolic transformations and the other one the other case what is that consisted only of hyperbolic transformations. So, pi 1.

So, the second case is when the deck transformation group. So, the point is that this is this is isomorphic to z or 0 the fundamental group has to be the z or 0 that is what we proved last time and so if the deck transformation group consists of parabolic transformation; parabolic elements mind you; these are Moebius transformations that fixed the at that or automorphisms of U. So, they are Moebius transformations in here and they have one fixed point in the whole extended complex plane right. So, in this case what we found was that the universal covering X sub univ to which is U to X p can be identified with U to delta star.

So, there is an identification like this. So, there is a bi-holomorphic map like this, there is a biholomorphic map like this and this map is just given by z going to e power 2 pi I e power 2 pi I z. So, this is the; what you get if the deck transformation group contain only parabolic elements and of course, once you identify the universal covering with this, then the deck transformation group of this covering is the subgroup of; is identified with the subgroup of translations by a single non-zero complex number which in this case we have taken as one alright. So, you can see that this is; this delta star is just U modulo z, it is delta star is just U modulo z and of course, z is a fundamental group of delta star is a isomorphic to the fundamental group of z delta star.

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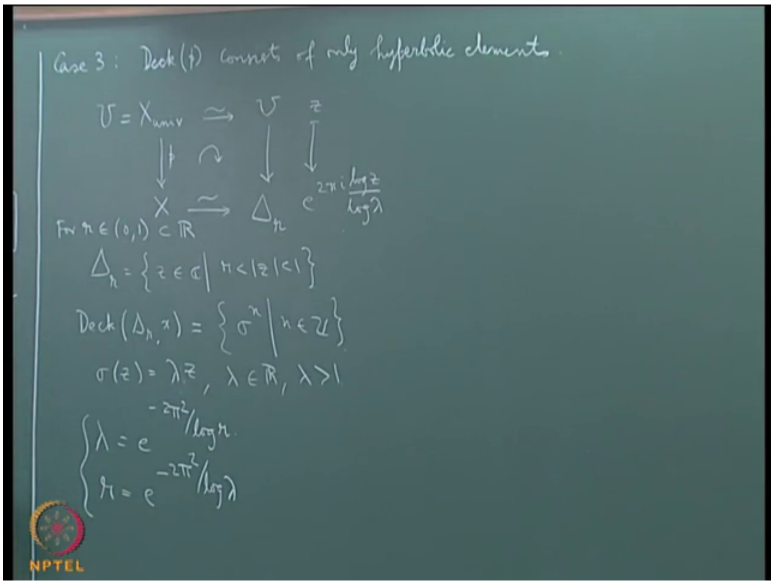

So, we saw this case then we looked at the other case when the deck transformation group consisted only of hyperbolic elements. So, deck p consists of only hyperbolic elements. So, there is there is an assumption that I have forgotten to say of course, I am assuming one more important thing that I must recall; I am assuming that the fundamental group of X are Abelian. So, I should also write that somewhere here assume. So, let me write this here assume pi one X comma X is Abelian; this is a very very important assumption; it is only because of this assumption that you get these 2 distinct cases because you because you know that a parabolic element cannot commute with the hyperbolic element.

So, the moment one element is parabolic and the deck transformation group is commutative because its isomorphic to the fundamental group then every other element has to be parabolic and the moment the deck transformation contains group contains one hyperbolic element again all other elements commute with these hyperbolic elements will force that all other elements are also hyperbolic. So, the important thing is you get these 2 distinct cases because you assume pi 1 is Abelian and in this case, what happens. So, in this case, it was a; there was something that one needs to check which I will try to do in this lecture. So, you see you have the deck transformation group. So, you know you identify X with delta sub r. So, we can identify the whole this whole covering map with the following map and what is this following map. So, let me write this. So, you know may be I use. So, let me not use. So, let me let me tell you what delta r; delta r is the annulus; the annulus with outer radius one inner radius small r; where small r is a real fraction. So, delta r is set of all z in c such that r is strictly less than mod z strictly less than one and this is for r in 0 1; the open unit interval in r real numbers delta r is this annulus alright and you see the map here is z going to.

So, let me write out the formula it is z going to e power let me check for a minute. So, is it z going to e power 2 pi I e power 2 pi I log z by log lambda this is the universal covering map where what is this lambda. So this, the deck transformation group the deck transformation group of X, I mean the deck transformation group of the cover. So, let me write delta r and of course, I fix a point let us fix a point say X and delta r right the deck transformation group turns out to be the exactly the infinite cyclic group generated by a single hyperbolic transformation and that is given by a z going to lambda z where lambda is greater than one lambda is a real number right.

So, this is how we had deck transformation group it was just well. So, it may like this; this is the set of all sigma power n where n is the integer and sigma power n is just sigma composed with itself n times and of course, sigma power 0 is to be taken as identity and sigma z is just lambda z lambda times z is just multiplication by lambda where lambda is lambda is real lambda greater than 1 and this lambda is related to this r in the following in the following way lambda is just e power. So, there is a formula for lambda its e power minus 2 pi square by log r there of course, I mean log r natural right and of course, this it this; this think and answer we written the other; we can write r equal to e power minus 2 pi square by log lambda 1 can write it in either way, alright, it is one and the same right.

So, this is how the universal covering looks I mean this is how the covering looks if X is Riemann surface with deck transformation group containing only hyperbolic elements and isomorphic to z right and so, I was making this remark in the last lecture that you know if you take different rs, then the corresponding annuli; they are not biholomorphic. So, that is what I would like to explain why. So, that is what I would like to do.

So, let me do that.

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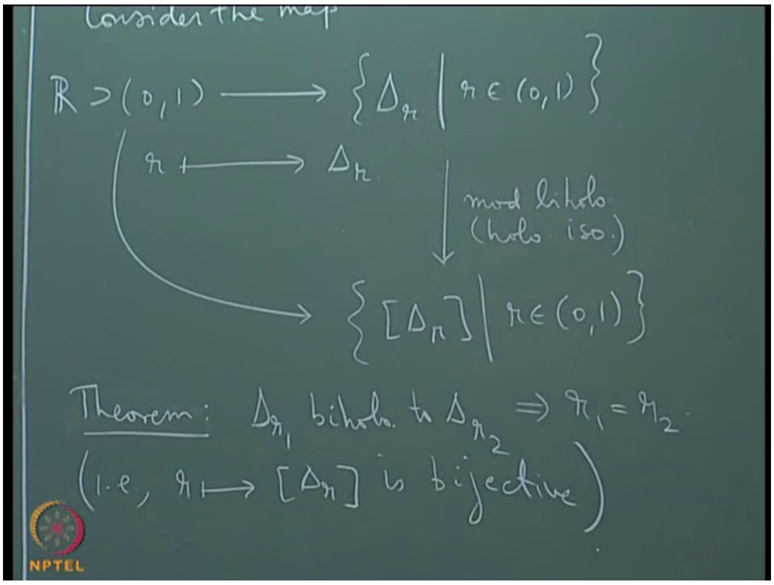

So, what I am going to do is. So, consider. So, you have this map consider the map. So, from 0 1 which is inside R to well the set of all possible del of all such delta r. So, may be I write such that r belongs to 0 comma 1, this is the set of all these annuli mind you all these annuli are open subsets of the complex plane. So, the automatically inherit the complex structure making them to Riemann surfaces and of course topologically the fundamental group is just fundamental group is just z.

And what is this map this is just sending r to delta r this is the natural map. And well by definition this map is a bijective map this is by definition a bijective map and what I am more going to do is I am going to follow this sub; I am going to go mod biholomorphic biholomorphic biholomorphisms that is holomorphic isomorphisms. So, what I will get here is, I will get; I will put this square bracket here where this square bracket of delta r denotes the holomorphic isomorphism class.

So, send every delta to its holomorphic isomorphism class and well you see therefore, I get map like this, I get a map like this and claim that the delta r is all are different for different r is a same as saying the map is actually bijective see this map is see this map is already surjective because this is my this is the bijective map and this is a surjective map and therefore, the composition is certainly surjective if you say that delta r 1 isomorphic to delta r 2 implies r 1 equal to r 2, then you are just saying that this map is also injective.

So, the claim is that r or rather the theorem is that delta r; delta r 1 biholomorphic to delta r 2 implies r 1 equal to r 2 in other words r going to biholomophic that is holomorphic isomorphism class of delta r is bijective. So, this is the claim right. So, we can we can prove this by using our technique of using covering spaces so as follows. So, what we will do is. So, let me draw another diagram. So, what we will do is let me start with; I will start with an isomorphism holomorphic isomorphism delta r 1 to delta r 2 and I will prove that r 1 equal to r 2 alright I will do that.

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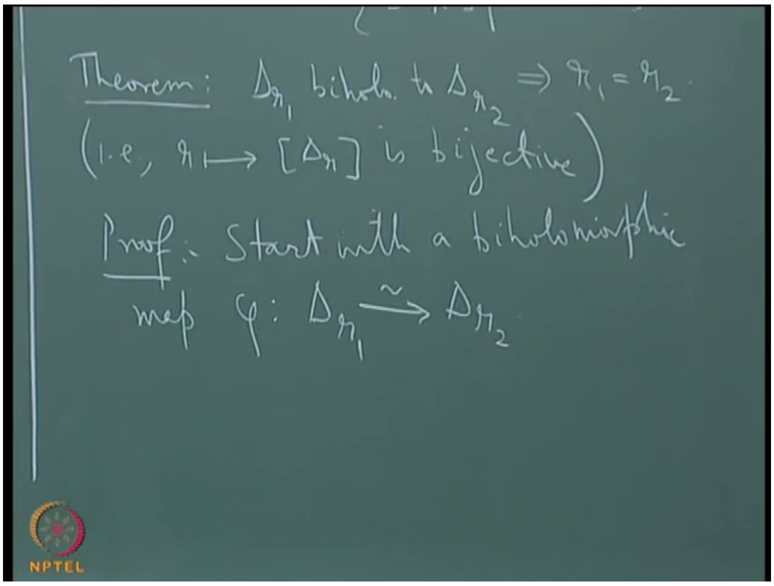

So, prove is start with a biolomorphic map phi from delta r 1 to delta r 2; start with a biholomorphic, we will have to prove that r 1. So, it is a bijective holomorphic injective holomorphic map which is also surjective; so the inverse also holomorphic. So, holomorphic isomorphism and I will have to deduce from this that r 1 is equal to r 2 and how does one do that how does one do that.

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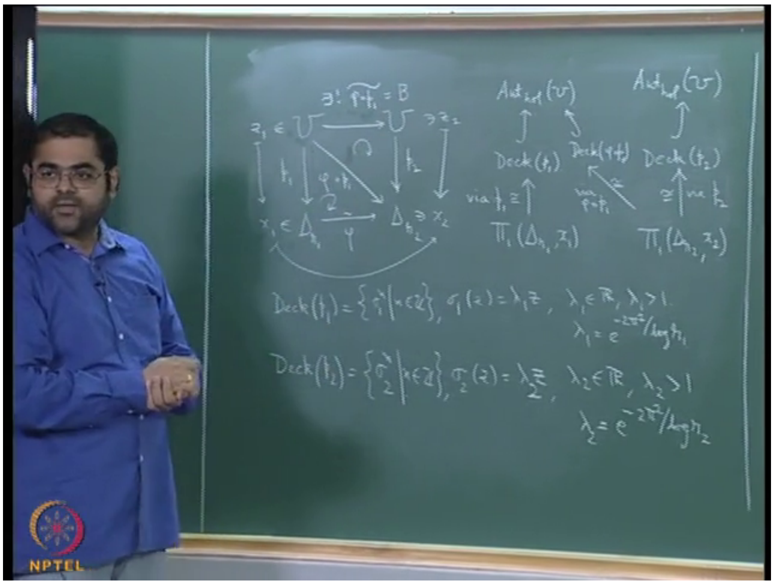

So, let me draw this kind of diagram. So, you see I have U which is a universal covering; I have let me call this as p 1 covering projection for delta sub r 1 and on the other hand I also have U I have p 2 which is a covering projection for delta r 2 alright and I have this isomorphism which is phi now compose. So, what I will do is let me fix a point z 1 in U alright and let us assume that z 1 goes to say x 1 here, right and let us; let me take; let phi take x 1 to x 2. So, phi will take x 1 to x 2 which is a point here alright and I will fix a point z 2 in U which goes to x 2.

So, you look at the order of things that I am doing the order in which I am doing this; you know reason I am doing this is because I need to identify the fundamental group below with deck transformations above. And so I have to fix a point below and so, if you fix the point here I get a point there and I have to fix points above which go to these points of course, I do have some freedom in choosing these points because I could be more I could be more than one choice there are more than there is more than one choice for this has more than one choice for this, but let us fix one choice here and one choice there alright.

Now, what is what is going to happen at the level of fundamental group. So, you see first fundamental group of delta r 1 based at x 1 you see that is going to be identified via via this map p 1. So, let me write down via p 1 because of the covering p 1 with the deck transformation group of p 1 and the deck transformation group of p 1 is well is sitting inside the holomorphic automorphisms of U right. So, this is what you get this is a diagram that you get corresponding to this cover alright then you also have similar diagram for this from the first fundamental group delta r 2 based at x 2 to you have an isomorphism group isomorphism because of the covering p 2.

And this is going to identify the deck transformation group with deck this is going to identify the fundamental group below with the deck transformation group of p 2 which is again holomorphic automorphism I mean subgroup of holomorphic automorphisms of U alright and well what is notice I mean going by what I have given here the basically the deck transformation group the deck transformation group of p 1 is just given by sigma 1 power n where n is an integer and where well sigma 1 of z is lambda 1 of z alright and lambda 1 is real lambda 1 is greater than 1 and of course, r 1 is related to lambda 1 by this formula.

So, lambda 1 is e to the minus 2 pi square by log r right and similarly I have deck formula for the deck transformation group of p 2, it is going to be generated by sigma 2. So, it is all possible compositions of sigma 2 with itself n sigma 2 to the n where sigma 2 of z is lambda 2 times z lambda 2 again in r lambda 2 greater than one and lambda 2 given by e to the minus 2 pi square by log r two. So, this is what we get for corresponding to these 2 covers right.

Now, what you must notice is that if I take if I take this covering map and follow it by this isomorphism I get another map like this and what is this map this is p 1 its p 1. So, let me write it here its p 1 followed by phi and this diagram commutes and this continuous to be a covering map this is still a covering map this is still a covering map because this is just; this is a biholomorphic holomorphic isomorphisms and this is a covering map. So, the composition is still a covering map alright and now because of this covering what is going to happen is that the fundamental group of delta r to delta sub r 2 based to x 2 is going to get identified as a subgroup of holomorphic automorphisms of this U o; namely it is going to get identified with the deck transformation group of this cover. So, I will have one more.

So, I will have also something like this, I will have an isomorphism here identification here. Now this identification is because of this cover because of this circle p 1. So, this is via phi circle p 1 and this will identify this with the deck transformation group of phi circle p 1 which is also going to be a sub group of holomorphic automorphisms of U. So, you have to keep. So, the way the deck the way the fundamental group is identified with the deck transformation group depends on the covering. So, for this covering you have an identification of this fundamental group with the deck transformation group of this covering which is which is which is a sub group of holomorphic automorphism surface the point is that these 2 are sub groups of same these 2 are being considered as subgroups here, alright.

So, the first thing I want to make the statement I want to make is you see there is a you know cover covering maps have a unique lifting power. So, you see I have this map phi circle p 1 that is a map from U 1 to this. So, I can lift this to a map like this. So, I get a map there exists unique lift which I call it as phi circle p 1 tilde that is this phi circle p 1 tilde is lift of this map to U and of course, it is a lift. So, this diagram commutes alright and this; this lift will be an isomorphism this lift will be an isomorphism that is quite easy to verify this lift will be an isomorphism. In fact, you can check that it will be a holomorphic lift and it will be injective.

So, phi circle p 1 tilde will be a holomorphic isomorphism from U to U and since it is a holomorphic isomorphism from U to U, it is going to be a you know it is going to be a Moebius transformation, it is going to be a Moebius transformation, it is an element of U the holomorph; an holomorphic automorphism of U, I will call this as B let me call this Moebius transformation as b. Now you see let us look at only this part of the diagram let look at only this part of the diagram and write again. Let us again analyze this of course, you know I could have anyway. So, let me rewrite it. So, I have the following thing I have pi 1.

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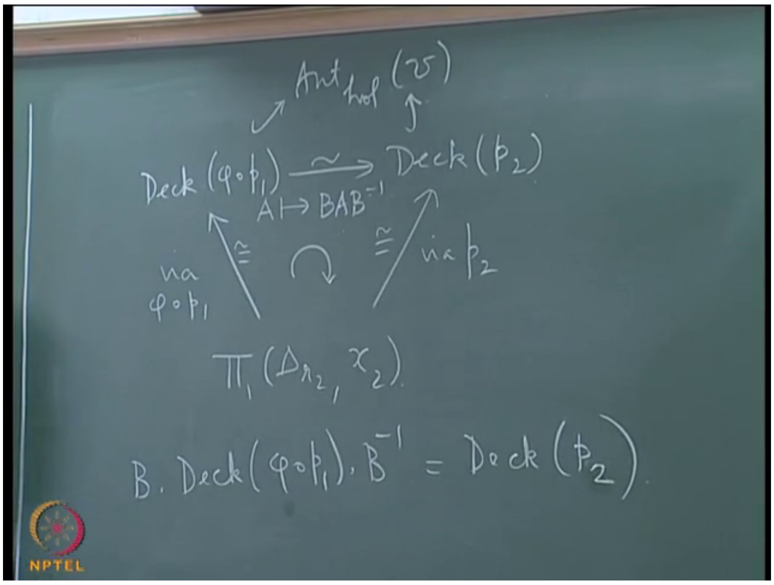

Delta r 2 based at x 2 and that is. So, I have this isomorphism this identification this is via phi circle p 1 this is identified with the deck transformation group of phi circle p 1 alright and well I also have the another identification this is via p 2 its identified with the deck transformation group of p 2 well and both of them are sitting inside the holomorphic automorphism and then see this commutativity of this diagram the commutativity of this diagram; what it tells you is that you know if you give me deck a deck transformation here if you give me a deck transformation of this cover how do I produce a deck transformation of this cover it is just by conjugating by B namely if you give me a deck transformation of this covering then. So, its a its a it is a it is a its an automorphism of U which respects the it which respects the fibers right and how do I get even a automorphisms of U which respects these fibers how do I get an automorphisms of U which respects this these fibers what I do is I apply B inverse I apply that automorphism then I apply B that is how I get an automorphism of this covering from an automorphism of this covering.

So, all I am trying to say is that there is a map like this which is an isomorphism and this map is simply give me any deck transformation for this cover you get a deck transformation of this cover simply by conjugating by b. So, you have a map a. So, this map is just a going to B A B inverse. So, in other words what I am trying to say is that I am just saying that and this of course, commutes this diagram of course, commutes and this is just the statement that you know if you change a covering by an isomorphism by an isomorphic covering then the deck transformation group will change by a conjugate in the group of holomorphic automorphisms of the universal cover, alright.

So, what this tells you is that this tells you that B that deck phi circle p 1 dot B inverse which is the image of this is actually deck p 2 that is what this is you get this, but what I want to say is I just want to say that this is exactly the same as the deck transformation group of p 1.

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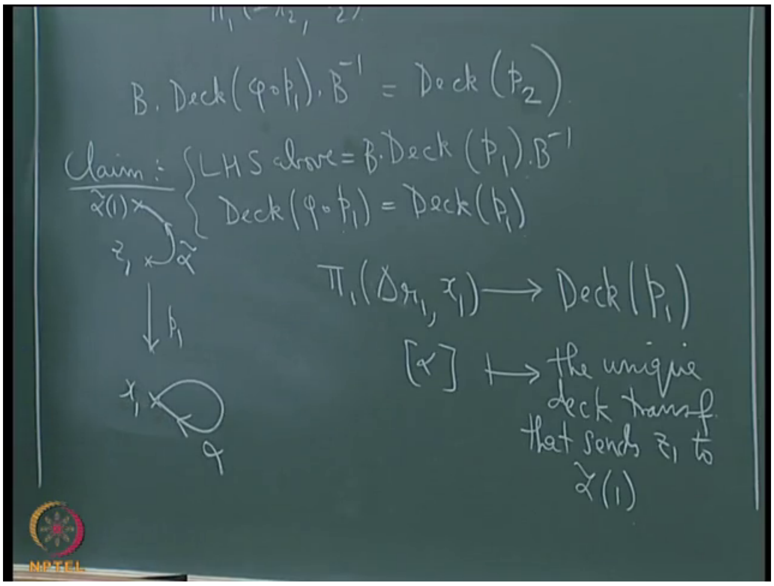

So, the claim is LHS above is exactly the same as the deck transformation group of p 1 and why is why is that. So, that is because of the way the identification of the deck transformation of the fundamental group is because the way you identify the fundamental group with the deck transformation group. So, let me explain that see I guess I should put; in fact, I should put B dot this dot B inverse and my claim is in fact, actually the claim is that this group deck phi circle p 1 is the same as deck p 1 that. So, what I am just saying is that the deck transformation group of this cover and of this cover are the same I am saying that deck transformations group are the both I am saying these 2 these 2 are equal actually? These 2 are equal here.

So, how does one do that So, if let us try to recall the way we got the deck transformation so what we did was you see I have see I had this point I have this point X one. So, this is the point s x 1 and here is a point z 1 above alright and what was the; what was the map from the first fundamental group of delta; delta sub r 1 based at x 1 to deck transformation of p 1 what was this map. So, you see this is this is the map p 1 alright and what was a; what was the what was this map. So, you see I took homo topic class of a loop alpha B starting and am at and ending at x 1. So, I took something like this and then what I did was I took it is; so, this alpha I took its unique lift above I got a unique lift alpha because of unique lifting property and that gave me new point here that point also lying was a point lying over x 1.

So, I got U point alpha tilde of one and it was the deck transformation that this alpha defined was unique deck transformation that takes z 1 to alpha tilde of one that is how it was find and there is one and only one. So, this is just the unique deck transformation that sends z 1 to alpha tilde of one alpha tilde being the unique lift of alpha.

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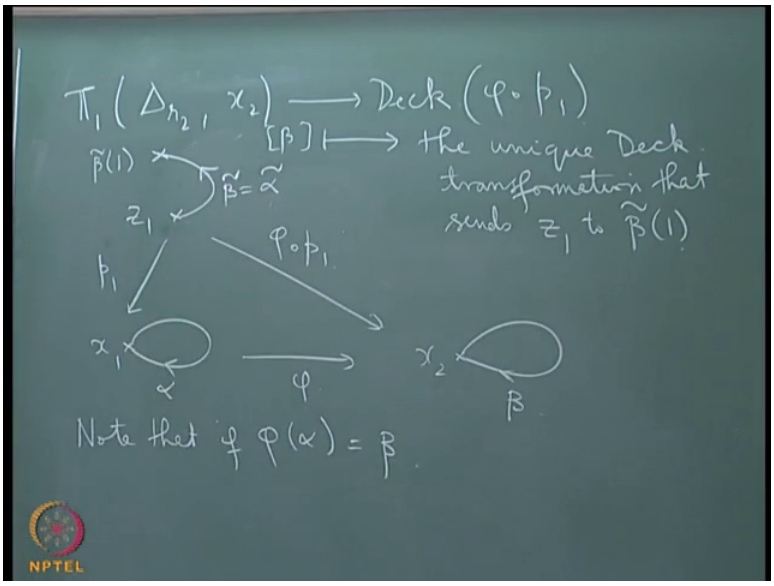

Now, in the same way how do you how do you get an identification of pi one of this with the deck transformation group there let me write that down pi 2 pi 1 of delta r 2 comma x 2 to the deck transformation group of phi circle p 1; how was this done this was done in the following way namely what you did was well you had you had x 2 a point in delta r 2 alright and then you had this point z 1 and z 1 is being mapped to x 2 by this by phi circle p 1 because p 1 takes z 1 to x 1 then phi takes x 1 to x 2 alright and what do we do; well, this is that the map here is defined by well you send beta to the unique deck transformation that that that sends z z 1 to beta tilde of 1.

So, namely what you did was you took beta here homo topic class and then you well lifted it to beta tilde. So, let it cramp; let me try to draw slightly better diagram. So, here is z 1 and I had this beta tilde which is the lift unique; lift of beta starting at z 1 for this map for the covering phi circle p 1 and this ended at a point this point is also going to lie over x 2 for this cover and this point is beta tilde of one and the deck transformation you are getting the automophism of the upper plane you are getting is the one that is is the unique one; that is sense z 1 to beta of tilde of 1.

Now, notice that you know note that if phi of alpha is beta namely I will put in this diagram here. So, there is also there is also x 1 here and there is also alpha here suppose alpha goes to beta under phi under the isomorphism p, then it is very clear that the lift alpha alpha alpha; the lift alpha tilde of alpha under this map which is p 1 is the same as beta beta tilde which is a lifting under this map after all well alpha tilde is the lift of this and therefore, it is also be lift of beta alright. So, beta tilde and alpha tilde are one and the same right. So, the moral of the story is that you know the image of this inside the holomorphic automorphisms of U which is the deck transformation covering of p 1 is exactly the same as the image of this in the in the holomorphic automorphisms of U.

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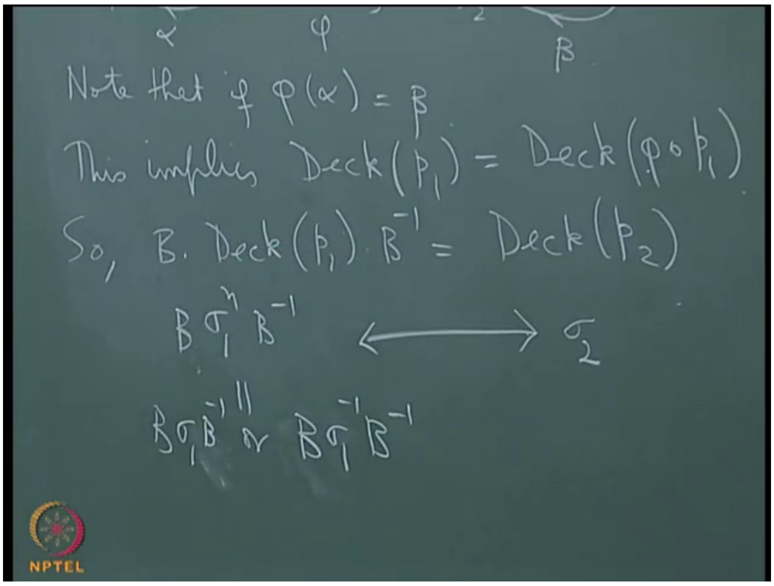

So, so this implies this implies that you know the deck transformation of the deck transformation of p 1 are the same as the deck transformations of phi circle p 1 deck transformations as the these 2 groups are one and the same inside the group of holomophic automorphism of U well. So, that proves the claim.

Now, therefore, you see. So, what I can do is in this here I can replace this by deck of p one. So, finally, what I get is deck p 2 is conjugated of deck p 1 right by b. So, deck p dot deck of p 1 dot B inverse is the same as well deck p 2 these 2 are one and the same. Now you see you take you take this element sigma 2 this is generator B 2 this has to correspond to an element here and that element here it has to be of the form B one sigma sorry B sigma 1 power n B inverse that is how elements here look like because elements in deck p 1 look like composites of sigma with itself.

So, that you see this is a this is the group identification alright and the generator only has to go to a generator therefore, this has to be either sigma 1 or it has to be sigma 1 inverse there are only 2 possible generators it is an infinite cycle group isomorphic to z there are only 2 possibility generators. So, this has to be sigma 1 or sigma 1 inverse. So, the moral of the story is that the up short of all this is that either sigma 2 is equal to sigma 1 or sigma 2 is equal to sigma 1 inverse that is what you get now what does that lead to. So, let me let me write here I think I can rub of this continue.

So, well I will get; so, you see I will get; so sigma 1 or sigma 1 inverse is sigma 2, this is what I get, alright and well I think I should be a little careful I should either say I should put I should conjugate by B, I should be careful B sigma 1 inverse or not exactly sigma 1 or B sigma 1 inverse, I should be careful, because if sigma is generator of this, then B sigma B inverse is generator of the conjugated mode. So, let me make that correction.

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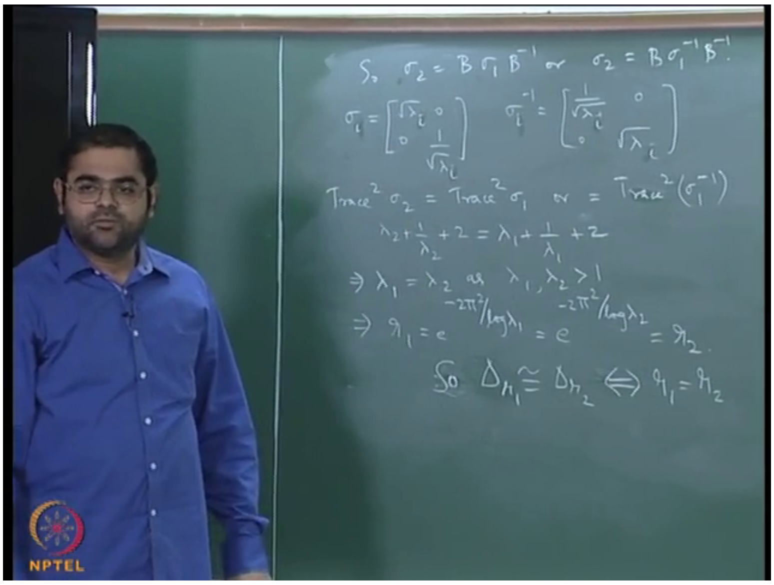

So, sigma 2 is either B sigma 1 B inverse or sigma 2 is B sigma 1 inverse B. Now what is sigma 1; see sigma 1 if I write it as a matrix alright it is sigma 1 of z is lambda 1 z. So, it will be root lambda 1 0 0 1 by root lambda; this is what sigma 1 is sigma 1 sigma one inverse is going to be well 1 by root lambda 1 0 0 root lambda 1; this is how I write representative in s l 2 alright and well the same works for 2 also. So, I will just put subscript i.

So, I just put I here and now compute trace compute the trace. So, what you will get is trace square sigma 2 is going to be the trace square sigma 1 or is going to be trace square sigma 1 inverse depending on whether it is equal to this or that because you know under conjugation trace is not going to change if you write this. So, if you write this down what you will get is you will get if you take both cases you will simply get lambda 1 lambda 2 plus 1 by lambda 2 plus 2 is equal to lambda 1 plus 1 by lambda 2 plus 2 and e if you solve this alright what you will get is well you are going to get lambda 1 is equal to lambda 2 because both lambda 1 and lambda 2 are greater than 1.

So, what you will get is well lambda. So, this will imply lambda 1 is equal to lambda 2 as lambda 1 comma lambda 2 are both greater than 1; this what you will get and if lambda 1 is equal to lambda 2 you will get r 1 is equal to r 2 because r 1 is just e power minus 2 by square by log lambda 1 which is equal to e power minus 2 pi square by log lambda 2 which is equal to r 2.

Thus, we get that the inner radii of these annuli are one and the same and the beauty is to see that we have applied diligently covering space theory to prove this. So, this implies r 1 equal to. So, I should say delta r 1 isomorphic to delta r 2 imply if and only if. So, I will put so r 1 equal to r 2. So, this clearly tells you why 2 annuli with different rs; they are not going to biholomorphic to one another. So, I think that clarifies to some extent the remarks I made towards the end of the last lecture.

So, we will stop here.