An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1 -dimensional Tori and Elliptic Curves

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Lecture - 22
Characterizing Riemann Surface Structures on Quotients of the Upper Half-Plane with Abelian Fundamental Groups

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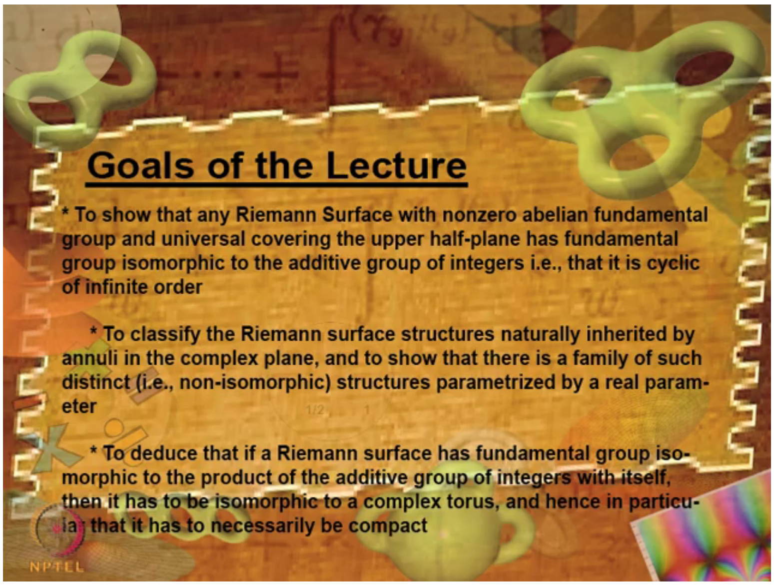

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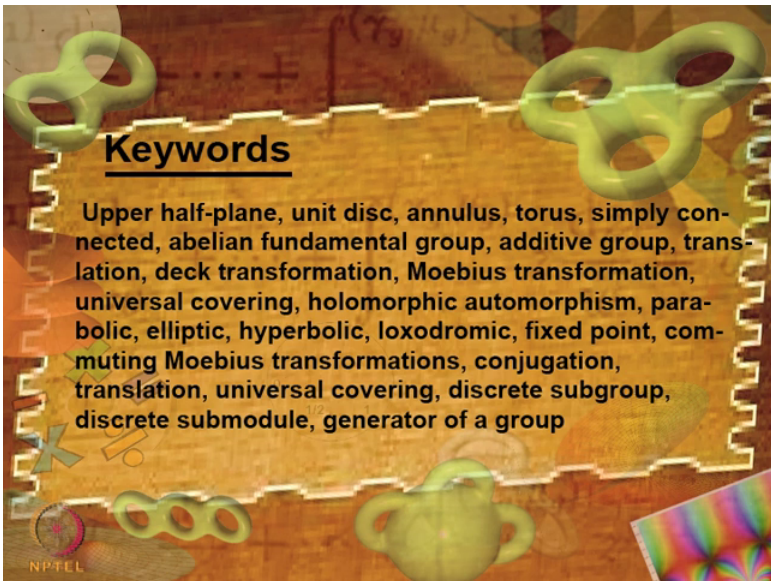

So let us continue with our discussion you see the last lecture we derived certain important properties of the fundamental group of Riemann surface, namely we found that the Riemann surface the formal group is torsion free, namely you cannot have any elements of finite order. And we also found that the only types of elements that you can get in the fundamental group or the Mobius transformations which are either parabolic or hyperbolic.

So, you cannot have elliptic elements in the fundamental group, you cannot have you know loxodromic non hyperbolic elements and of course, in all this we are identifying the fundamental group of the Riemann surface with choice of some base point, with the deck transformation group of the universal covering; and we must remember that the deck transformation group is a subgroup of Mobius transformations fine.

So, I will now have to go on with trying to get the classification of Riemann surfaces, which have universal covering the upper half plane and which have Abelian fundamental groups. So, that is what I am going to start with.

So, let me write that down.

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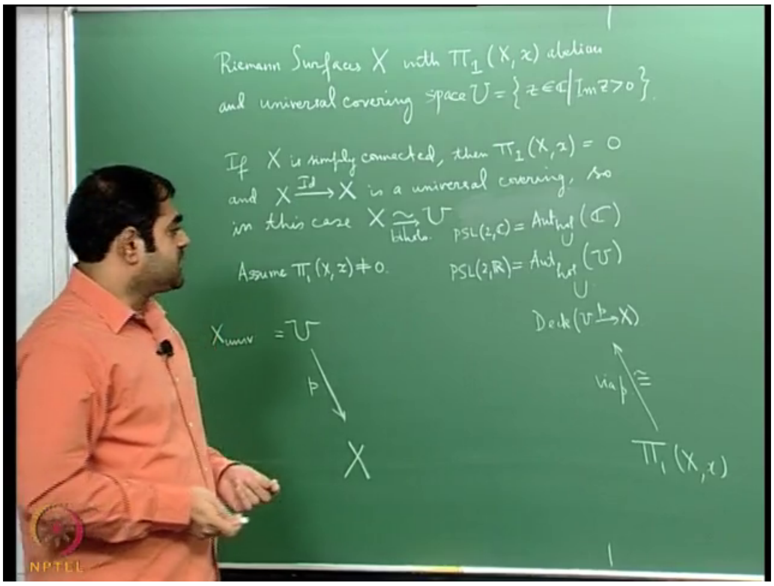

Riemann surfaces X with pi 1 capital X comma a small x Abelian, and universal cover covering space U which is set of all z and c is a real part I mean imaginary part of z is positive. So, we are looking at these types of Riemann surfaces of course, small x is a fixed point of capital X and of course I should also remind you that the upper half plane is bi holomorphic to the unit disk. So, well if you see first of all if X simply connected, then then pi 1 is 0 pi 1 of X comma x is trivial 0 and X to x identity map is a covering is the universal covering.

So, in this case X is bi holomorphic to U. So, we look at the situation when the fundamental group is nontrivial; so i. So, first of all, let me draw a diagram. So, you have well you have x here and so, I have the projection the covering projection this is X sub unit is U, this is X sub unit the universal covering space and well because of this you know if you fix a base point small x in capital X then the fundamental group below is identified with a subgroup of holomorphic automorphisms above and that subgroup is precisely the group of subgroup of deck transformations of this cover ok.

So, the corresponding diagram is well let me also draw it here. So, I will have to write across this board. So, I have pi 1 of capital X comma small x, there is identification. So, I will put this say it is a isomorphism, this is an identification via p that means, but because of this covering and this identification is with the deck transformation group of this cover, which is a subgroup of well the holomorphic automorphisms of the universal coverage space. And well I can go one step further and make you realize that this is sitting also in the holomorphic automorphisms with the whole complex plane.

So, if you want well it is maybe I remove this letter, these few letters so that I have some more space to write. So, let me remove this, so here is. So, this I write it as bi holomorphic and so this is sitting inside automorphisms the holomorphic automorphisms of complex plane. Mind you that this is identified with p s l 2 R, these are Mobius transformations with of the form z going to a z plus B by c z plus d with a B c d real numbers, and a d minus B c equal to 1 these are the representatives here, and of course, this is the full group of Mobius transformations which is PSL 2 C. So, this is all complex entries and this is a sub group of that right. So, that is because of this covering.

Now,. So, I am. So, assume. So, I am here I am assuming that the first fundamental group is nonzero well. So, the first statement I want to make is that the deck transformation group can contain only parabolic or hyperbolic elements; this is what we have seen last time. I mean elliptic elements are not allowed why they are not allowed is because an elliptic Mobius transformation will fix a point of view, and deck transformation a non trivial deck transformation is not supposed to fix any point otherwise it will become equal to the identity transformation.

So, elliptic transformation is not allowed, and we have also seen that for the upper half plane the condition of loxodromic is the same as it being hyperbolic Mobius transformation which is loxodromic and which preserves upper half plane has to be also hyperbolic necessarily hyperbolic ok. So, there are only 2 possibilities. So, let me write that down here because I want space for some more to extend this diagram.

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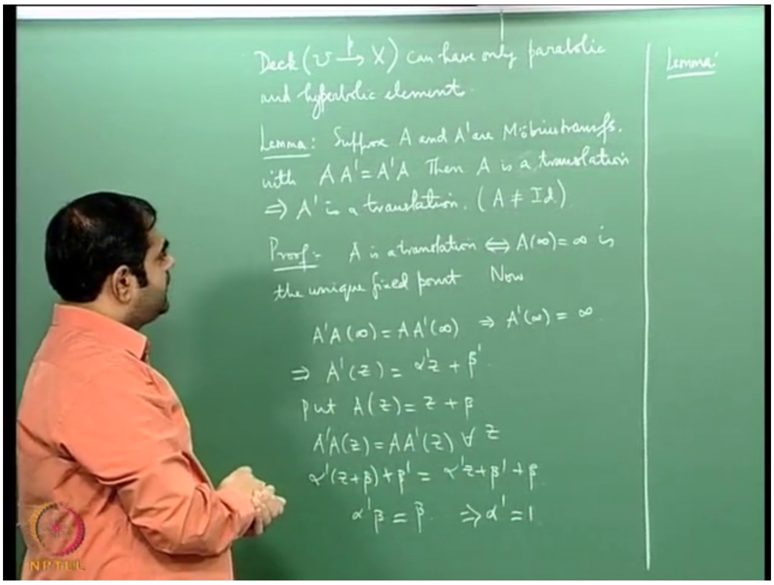

So, there are. So, deck U p X can have only parabolic and hyperbolic elements. So, this is what we have from the last lecture and you see I claim that because of this additional assumption that the fundamental group is Abelian, which means that the deck transformation group is Abelian after all the fundamental group is identified with the deck transformation group. I claim that because of this Abelian nature, which makes every deck transformation commute with every other deck transformation. This commutative the forces 2 mutually exclusive cases namely one in which all the elements of parabolic and the other in which all the elements are hyperbolic.

So, what I am trying to say is a parabolic and hyperbolic element cannot commute. So, we will see why that is true. So, let me write a few lemmas, suppose A and A prime or Mobius transformations with you know A A prime is equal to A prime A alright that is a and d A A prime commute. So, in A and A prime are a pair of commuting Mobius transformations then A is a translation is a translation implies A prime is a translation. So, I am just telling the very obvious fact that only translations can commute only with translations ok.

So, what is the proof? The proof for this is you see you know A is a translation means that A of infinity equal to infinity is the unique fixed point, if A is a translation then infinity is a fixed point and you know translations are parabolic. In fact, an element is parabolic Mobius transformation is parabolic if and only if it is conjugate to a translation. So, and if it is parabolic by definition it has only one fixed point. So, infinity is a unique fixed point alright. Now you see A prime A of infinity will be A prime infinity. So, what, but you see a of infinity is infinity.

So, this will tell you that A prime infinity has to be infinity why? Because you see what does this tell you? A of infinity is infinity. So, you get A prime infinity is A of A prime infinity; that means, A prime infinity is a fixed point for A, but the only fixed point for a is infinity therefore, A prime infinity is equal to infinity, but if A prime infinity is equal to infinity the near prime is a translation. So, this implies well we have to say a little bit more, this implies that A prime of z these are the form alpha z plus beta.

If infinity is a fixed point for a Mobius transformation, then it has to be of this form. Now you see put I am just trying to claim that alpha is 1. So, you put A as of z see after all A is a translation. So, you write as z plus well since it is A prime let me put primes here and write for A of z, z plus beta and write out the condition that A prime A of z is A prime of z for every z. So, you know A prime A of z is equal to A prime of z for all z what this will tell you is going to tell you that well; so if I. So, what does it been first apply a then apply A prime let us go into same as first applying A, A prime and then applying A to z. So, first apply A to z, I get z plus beta and apply A prime to that.

So, I get alpha prime z plus beta plus beta prime is equal to I get here I have to first apply A prime, which is if I apply A prime to z I will get alpha prime z plus beta prime and then I have to apply a which is just adding beta. Now if you compare both sides you will get that the beta prime are going to cancel and then I will get the alpha prime zs are going to cancel, I am going to get alpha prime beta is equal to beta, this is what I am going to get and mind you of course, I want to load the case that beta is 0. So, that I can cancel by beta and conclude alpha prime is 1. So, you see beta equal to 0 corresponds the case when A is identity and I am assuming I will let me assume that A is not identity, I am not I do not look at the identity transformation because that is not very interesting a not equal identity ok.

So, A is not equal to identity. So, beta is non zero therefore, I can cancel beta and I will get alpha prime is 1. If alpha prime is 1 then A prime z is z plus beta prime. So, it is also a translation alright. So, that proves this lemma alright now.

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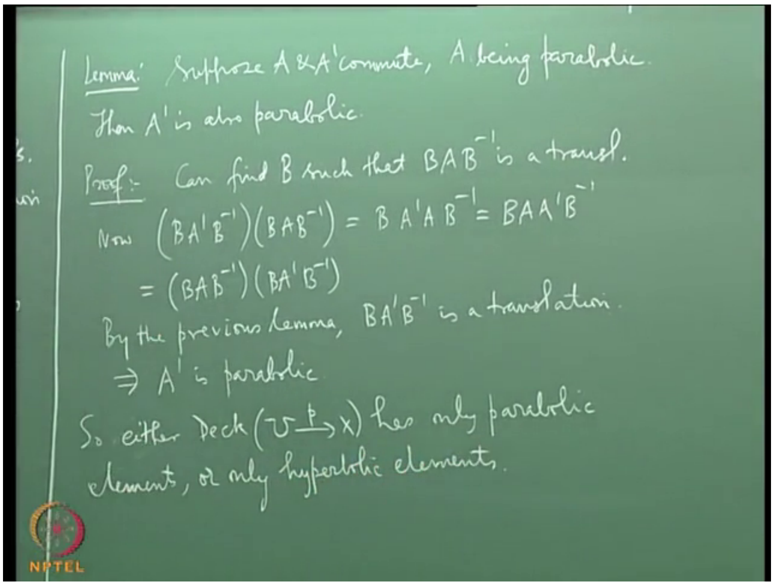

I will make another simple lemma. So, what this tells you is that if a Mobius transformation commutes with the translation then it has to be a translation. Now, but you know translation is just up to conjugate translation is just a parabolic element. So, I am saying the same thing will happen for a parabolic elements also, namely if a Mobius transformation commute so the parabolic element it has to be parabolic. So, let me write that lemma.

Suppose A and A prime commute a being parabolic then you know A prime is also parabolic alright. So, what is the proof? The proof is well you see a is parabolic. So, by our characterization of parabolic transformations, you can find a B such that conjugating A by B makes it to translation. So, can find B such that B A B is the inverse is a translation, and well again in all these cases you know the case when one of the transformations is the is the identity transformation, I am not at all worried about that case because that is the uninteresting case ok.

So, well B A B inverse is the translation, but now notice so; that means, what? B A B inverse has unique fixed point infinity means a translation. Now you see B A prime B inverse, times B A B inverse this will be just B A prime A B inverse, but this is B A A prime B inverse because A prime A is a A prime and then I can write it as B A B inverse B A prime B. Inverse in other words I am just saying that because A and A prime commute, do their conjugates by B. Now look at lemma 1 lemma 1 says that if 2 Mobius transformations commute and one is a translation then so is the other.

So, by the by preceding lemma by the previous lemma, what will happen is that B A B inverse is a translation. So, B A prime B inverse is also translation and you see if B A prime B inverse is translation then of course, A prime is parabolic. So, the moral of the story is that if the deck transformation has one parabolic element then every element is parabolic I mean. So, it contains only parabolic elements. So, either deck U p X has only parabolic elements or only hyperbolic elements. So, the moral of the story is that you have 2 distinct cases mutually exclusive cases; one in which the deck transformation consists of only parabolic elements the other one is the case when the deck transformation consists only of hyperbolic elements ok.

So, let us assume that the deck transformation group consists of only parabolic elements.

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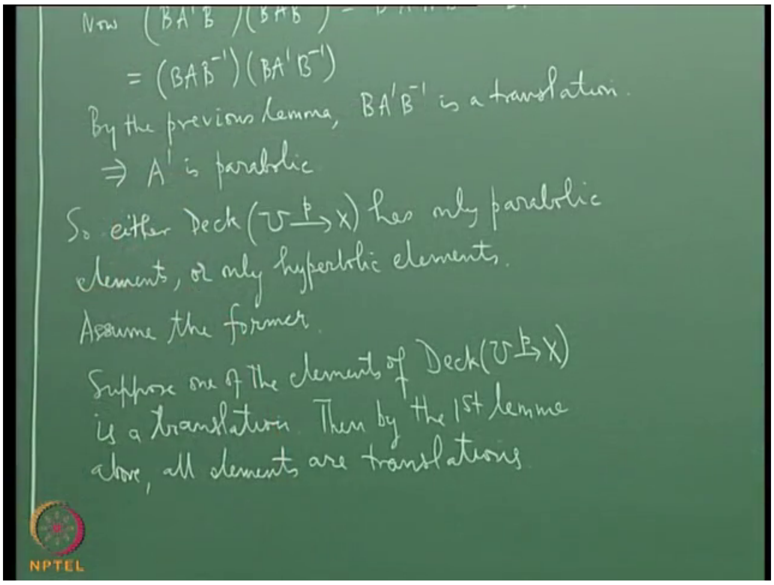

Assume the former namely the deck transformation group has only parabolic elements. So, here again I distinguish between 2 cases just for clarity, suppose one of the elements of the deck transformation group is a translation. See after all deck transformation group contains parabolic elements and if necessary I can make a conjugation and make a parabolic element into a translation. But even without the necessity of that suppose there is one element namely a parabolic deck transformation is already a translation. Then you see again this lemma will tell you that all other elements are translations, and then by the first lemma above all elements are translations, ok.

Now, you see so what happens in this case is well.

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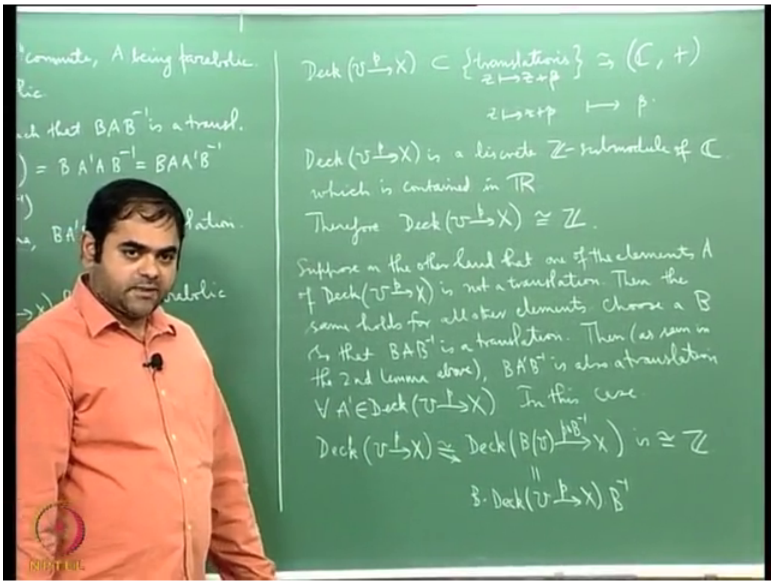

You see you have deck transformation groups is well; it is identified with a subgroup of translations. So, translations are the form z going to e z plus beta, this is a group under addition translations I mean if you I mean this is a group at a composition of mappings. And if I identify z going to z plus beta with say the complex number beta, then this is identified with complex numbers and this is well this is a group at addition. So, the translations are Mobius transformations under composition and identifying z going to z plus beta with beta; the complex number beta will identify this Abelian group of translations, under composition with the additive group of complex numbers now notice that.

So, I again want to make the same statements that I did when we were trying to classify Riemann surfaces with universal covering space the complex plane and with nonzero fundamental group. If you go back and recall what happened at that time, we found that the fundamental group turned out to be certainly Abelian that was because just like in this case we were able to identify the deck transformation group as a group of an Abelian group under addition and subgroup of translations. And then you know there we made the statement that the deck transformation group is a discrete z module, and the discreteness came because of the fact that this is universal covering and these are deck transformations ok.

And then we gave the statement that a discrete z sub module of c has to be either generated by one integer multiples of one nonzero complex number, in which case it is isomorphic to zee or it is generated by integral multiples linear integer linear combinations of 2 nonzero complex numbers with non real ratio, in which case it is isomorphic zee cross zee. So, in the by the same kind of argument, what you can see is that the deck transformation group is a discrete Z sub module of C.

So, this argument the proof of this is exactly as it was for the case when the universal covering was the complex plane alright there is no difference in the proof. But there is something new now; you see notice that these are all holomorphic automorphisms of U. Since this are holomorphic automorphisms of U I want you to remember that the entries beta that you get here, that is a image of this there is going to land inside r. The elements here are certain holomorphic automorphisms the upper half plane, but holomorphicauto morphisms the upper half plane are Mobius transformations with you know with of the forms that going to a z plus B by c z plus d with a b c d real. So, the point I want to make is that beta is real. So, what you actually get is a discrete z sub module of c, which is contained in r which is contained in r ok.

So, you know now we have this now let me recall that lemma. I discrete z sub module of c can have only three possibilities either it is 0, zero sub module or it is isomorphic to zee and it is integral multiples of a single nonzero complex number, and the third case is it is isomorphic to zee cross zee and it is integer linear combinations of 2 nonzero complex numbers, but these 2 nonzero complex numbers are supposed to have non real ratio; that means, they are supposed to be linearly independent overall. But you see if it is contained in R, I cannot find 2 entries which are linearly independent over R itself. So, therefore, the only possibility is that it has to be isomorphic to zee.

Therefore, the deck transformation group is isomorphic to zee; the deck transformation group is zee there is no other choice and the isomorphism is given by. So, that means, there is a generator here, and the generator is a translation and every other translation is just applying these translation or it is inverse finitely many times that is all that you have in the group. So, the amazing thing is that in this case the deck transformation group is just isomorphic to zee fine.

So, I have taken. So, this is the case when one of the elements is a translation alright. Now suppose one of the elements is not a translation let me go to the other case, then of course, all the other elements are also not going to be translations because you know the translations commute only with translations alright. So, suppose on the other hand that one of the elements of deck is not a translation suppose it is not a translation alright? Then every other member is also not a translation because they all commute with this one because it is a commutative group. Then the same holds for.

So, let me call that element as A, suppose one of the elements A in the deck transformation group is not translation, then the same holds for all other elements namely all the others are also not translations. Again you can find a B. So, that B A B inverse becomes a translation, after all B is going to be a Mobius transformation that is going to take the fixed point of A to infinity that is all you need alright. So, find choose a B so that B A B inverse is a translation I can choose A B. So, that B A B inverse the translation, but you know if B A B inverse translation then be A prime B inverse is also a translation for every other element that is because B a all the B A prime B inverses they will all commute with B A B inverse ok.

So, then as seen in this second lemma above B A prime B inverse is also a translation for every A prime the deck transformation group. So, what is the situation now? The situation now is that again you will find the deck transformation group is isomorphic zee; only thing is that you have to shift to an isomorphic cover. So, how do you do that? Let me go back to this diagram here and write it down. So, you see I have this B you see what is this B.

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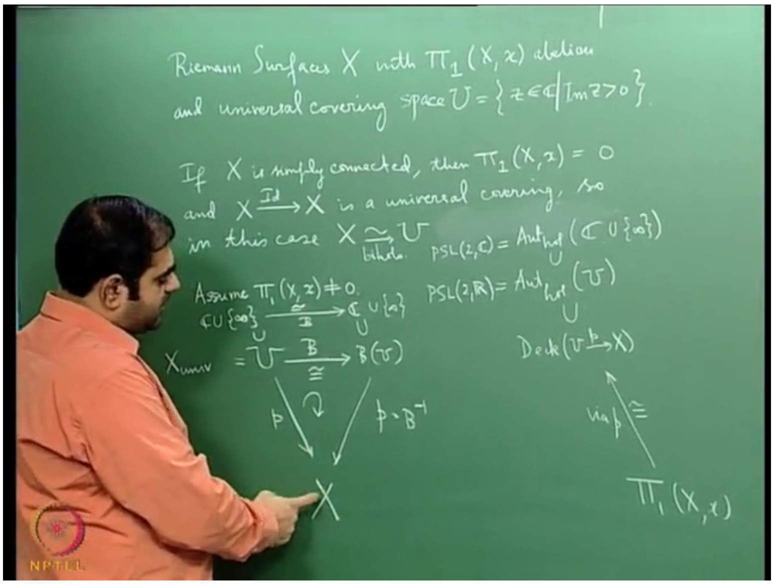

So, I have U sitting inside the complex plane the upper half plane sitting inside the complex plane. So, this is subset relationship, that I will do not confuse this with U. I am just writing U subset of C alright and then I have this B, B you see B is a Mobius transformation alright and B will take well B will take C to C. In fact, if you want maybe I should put C union infinity, because it may take a finite point to infinity; so I to take the whole.

So, here again I think I have made a mistake. In fact, I should write C union infinity this is of course, the whole group of Mobius transformations including the point at infinity. If you take only holomorphic automorphisms of C then it is always of the form it is upper triangular form. So, you want get PSL 2 C. So, please correct this small mistake. So, well I have U and this is B of U. So, B of U well U is in the upper half plane in the in C in infinity and B is an isomorphism of C union infinity C union infinity and it will take U isomorphically to B of U ok.

And if I take this map defined to B by the defined by the community of this diagram, it is first apply B inverse then apply p, then you see this is also covering after all it is an isomorphism followed by a covering. So, this is also a covering of X, this is also a universal and this also universal covering because after all B U is isomorphic to U it is bi holomorphic to U. So, it is also topologically homeomorphic to U. So, it is also simply connected. So, this is also another avatar of the universal covering for X. Now what does this what happens to this when I do it at the level of deck transformations.

So, what is going to happen is just like this covering identified the fundamental group here with the deck transformation group a subgroup of holomorphic automorphisms above, the same thing should happen here. So, what I am going to get is, I am going to get something like this.

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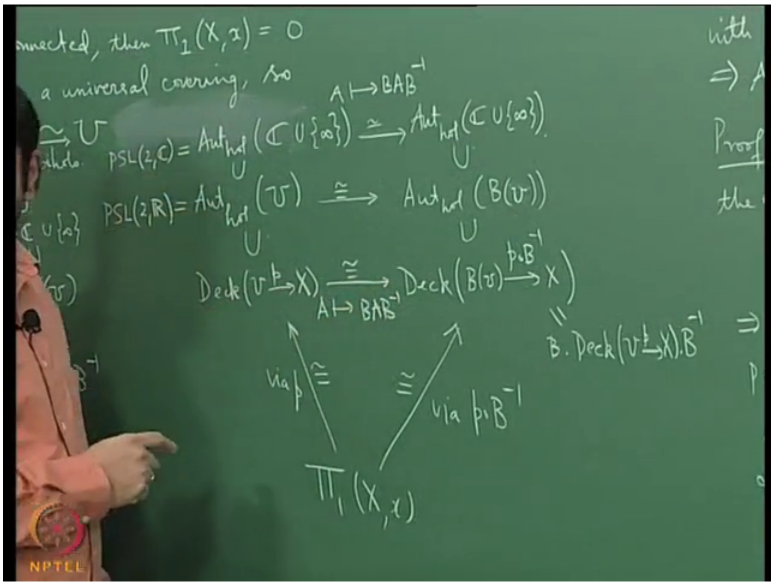

So, I will get another isomorphism identification this is via p circle B inverse and this is going to give me the deck transformation group of the other isomorphic cover namely B of U p sub to B inverse X and this is going to be well this is subgroup of the holomorphic automorphisms of B of U; B of U is some domain in the complex plane which is by holomorphic to U, alright.

And of course, this is again contained inside the well auto morphism group of all Mobius I mean the all possible Mobius transformations alright and what is this map? This map is just. So, how do you get a deck transformation here? Well I take a point here, I go by B inverse apply a deck transformation there and then come back. This is how starting with the deck transformation here I produce it deck transformation there.

So, what is the map here? The map here is this conjugation by B; it is just A going to B A B inverse. So, give me a deck transformation A here that is by definition a holomorphic auto morphism of U, which is going to preserve the fibers and how do I to what deck transformation does it go here it just it is conjugate. So, this is just A going to B A B inverse and it is the same map everywhere. So, there is an isomorphism like this, there is an isomorphism like this, there is an isomorphism like this in all these cases the map is just A going to B, A B inverse alright that is a map; it is just A going to B A B inverse.

So, here also it is a map A going to B A B inverse. So, this is the map I am looking at. So, let us go back to the current situation we are in, we have taken the situation that all the deck transformations are not translations, but then we conjugate by a single we pick any one deck transformation and you conjugate it with a single b. So, that the after conjugation it becomes a translation, then everything becomes a translation alright. So, what happens is that this?

So, what is this? This is see this is just B dot deck U p X dot B inverse, that is what is it is you are actually conjugating the whole group by B and well let me draw a line. So, that it does not and. So, in this case this group namely this now this is a subgroup of translations. And again it is going to be a discreet it can be identified with the discrete z sub module of C contained in R, it is going to be the same thing. As the earlier case and if that will go that it force that this is isomorphic to zee.

So, in both cases the deck transformation group is actually isomorphic to zee the integers, and it is generated by a single translation namely it consists of integer multiples of a single translation or rather I should say translations by integer multiples of a single nonzero complex number. So, in this case deck U p X is isomorphic to deck B of U, p circle B inverse X, which is equal to incidentally the conjugate of this it is just B dot deck U p X dot B inverse and this is isomorphic zee because again let me repeat it is discreet it is a discrete z sub module of C and it is going to land inside R it is a discrete set. So, as linearly independent elements over all it can contain only one, in which case you can get only zee. The only other the only other possibility is you can get zee directs some zee you cannot get zee directs some C ok.

Because you cannot find 2 translations here, you cannot find 2 real numbers which are linearly independent over R. R over R has dimension 1; any maximal linearly independent subset has only one element right. So, the moral of the story is that if your deck transformation group consists of only parabolic transformations, in that case the deck transformation group is just isomorphic to zee. Now I am going to say that literally the same thing happens in the other case; namely when the deck transformation group contains only hyperbolic elements let me do that. I think I will retain this diagram on this side right and I will kind of. So, select. So, what I am going to do is let me draw a line that.

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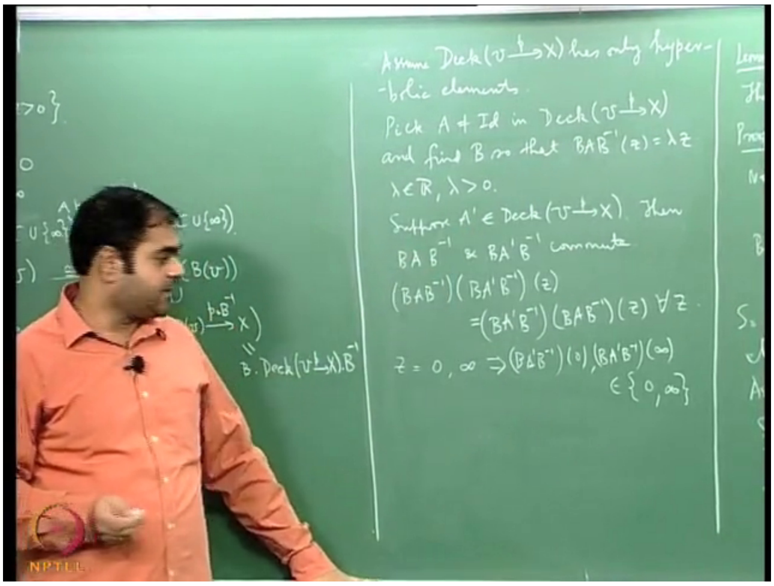

So, let us go to the other case; assume deck the deck transformation group U p X has only hyperbolic elements, assume it has only hyperbolic elements now. So, again you know a Mobius transformation. So, p let us pick 1 hyperbolic element and you know you take if you take a hyperbolic Mobius transformation, then you can conjugate it by a suitable Mobius transformation so that it is fixed points or of the form or 0 and infinity. So, it is of the form z going to lambda z, and since and if this Mobius transformation is going to preserve the upper half plane then you know lambda has to be real and lambda has to be positive.

Because any hyper any I mean that is the characterization of hyperbolic Mobius transformation. So, pick A article identity in deck transformation group and find B so that B A B inverse of z is lambda z, lambda is going to be real lambda is positive. So, this is possible because of our characterization of hyperbolic Mobius transformations right. Now I want to say that every if you take any A prime, because every other A prime here commutes with a I want to say that every if you conjugate the every other A prime with the same B, it is still going to be of the same form that is what I want to say.

So, suppose A prime is another deck transformation. So, mind you if A already has fixed points 0 and infinity then you can take B to B identity, you do not have to really look for a B different from identity alright. So, suppose A prime is a deck transformation is another deck transformation in this group, then again you see B A B inverse and B A prime B inverse commute.

We have seen that this is true because a and A prime commute. And if you look at B A B inverse of B A prime B inverse of 0 is equal to; let me put z. I am just writing the commutativity out and what I am going to do is I am going to put z equal to 0 you put z equal to 0, what I will get is, I will get B A B inverse of B A prime B inverse of 0 is equal to B A prime B inverse of 0 because B A B inverse of 0 is 0. So, what it will tell you and if I put if you put z equal to infinity, you will get B A B inverse of B A prime B inverse of infinity is equal to B A prime B inverse of infinity. So, what this will tell you if you put 0 z equal to 0 and z equal to infinity, you will it will tell you that B A prime B inverse of 0 and infinity are the 2 fixed points of B A B inverse. So, they are one among the two point zero and infinity.

So, this will tell you that B A B A prime B inverse of 0, B A prime B inverse of infinity, they both belong to 0 infinity they have to be one of these two. Notice that these 2 are going to be fixed points for B A B inverse and B A B inverse has precisely 2 fixed points Mobius transformation cannot have more than 2 fixed points, those 2 fixed points are 0 and infinity because of this form that we have assumed alright. So, now, we claim the obvious that you know B A B inverse B A prime B inverse of 0 is actually 0, and B A prime B inverse of infinity is actually infinity and not the other way round, that is where we will use the fact that A prime is actually hyperbolic.

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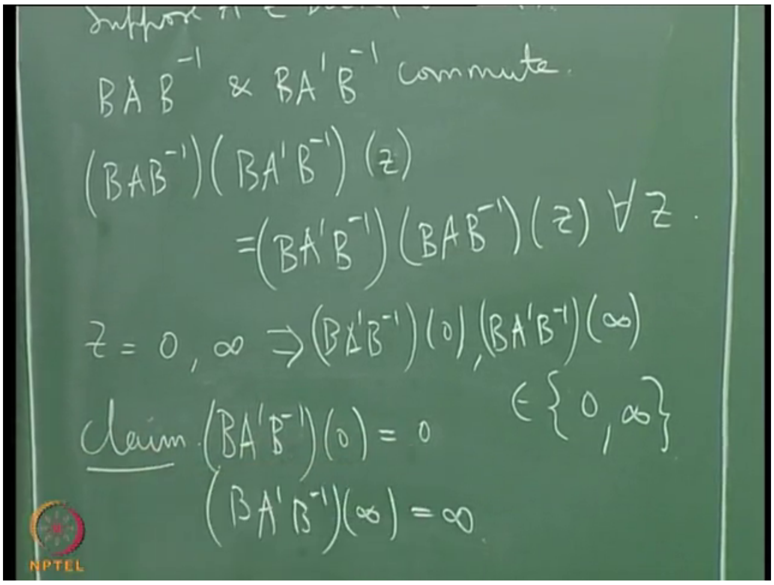

So, the claim is B A prime B inverse of 0 equals 0, B A prime B inverse of infinity is equal to infinity this is the claim alright. Well of course, they cannot be equal because you know Mobius transformations are one to one it cannot be equal. So, either these is equal to 0 and that is equal to infinity or the other way round, this may be infinity and that may be 0 if you assume that you get a contradiction. So, we can try to work that out.

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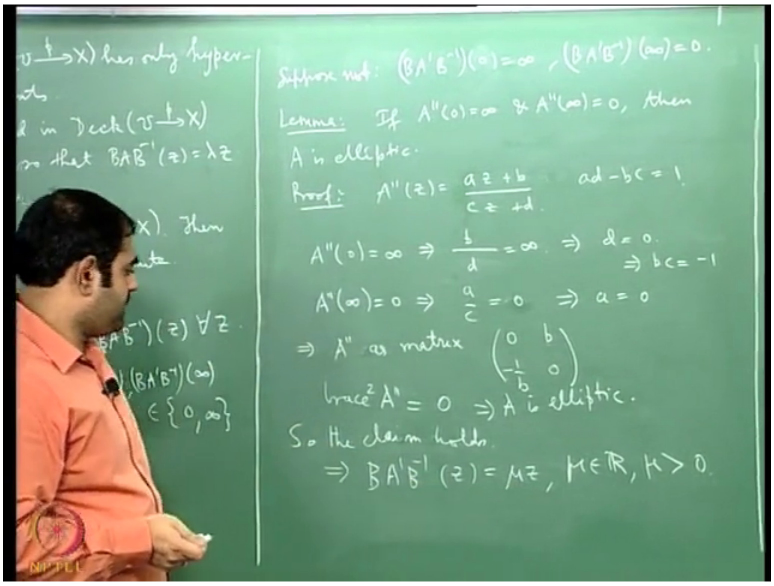

Suppose not. So, B A prime B inverse of 0 is infinity and B A prime B inverse of infinity is 0. Well you see I will work out a small lemma this must be obvious to you, but we can write it out if it A, A double prime of 0 is infinity and A double prime of infinity is 0, then A is elliptic because you see proof is you see write A double prime of z as a e z plus b by c z plus d with a d minus b c is equal to 1 write it in this form. Now A double prime of 0 is infinity. So, when I plug 0 inside this I should get infinity; that means, the denominator should vanish and. So, when I plug in 0, I get infinity. So, let me write that down I will get b by c z plus d is infinity, and you know this should force that c is 0 I will get b by d, I get B by d sorry I will get b by d and this will force the d is 0 alright. I cannot divide a complex number by another to get infinity unless the denominator is 0 ok.

And if d is 0 mind you. So, that will tell you already that b c is minus 1, and then A double prime of infinity is 0. So, when I plug in infinity if it is 0. So, the point is you divide both numerator and denominator by z. So, you can also add as if following a z is not zero you can write as a plus b by z divided by c plus d by z; if I plug in infinity the 1 by z terms will go off. So, I will get a by c. So, I will get a by c equal to infinity, I mean a by c equal to 0 that will tell me that a is 0. So therefore, what I will get is, I will get a double prime matrix as matrix is of the form well a is 0, b is as it is, c is minus 1 by b and d is 0 and you see this of course, determinant 1 alright, but what is it is trace squared? Trace squared is 0 that means it is elliptic. Trace squared a is 0 that implies a is elliptic. So, that proves this lemma.

So, now, if you look at this lemma then B A prime B inverse is 0 is infinity and B A prime B inverse infinity is 0 if that happens, then B A prime B inverse will become elliptic; that means, A prime is elliptic, but we have assumed A prime to be hyperbolic after all. So, it is not possible, the claim a holds. So, the moral of the story is that this claim holds and therefore, if you take any other deck transformation A prime, then B A prime B inverse is also of the forms z going to some mu of z alright where mu is and is hyperbolic. So, that mu is real and mu is positive. So, this implies B A prime B inverse of z is equal to some mu z mu real mu positive ok.

And of course, please remember that in all these cases lambda is not 1 unless A is the identity lambda is not 1 so if I have taken A not equal to identity, then this lambda cannot be 1 and if A prime is not identity then mu cannot be 1 right. Now mind you again I have the same kind of situation.

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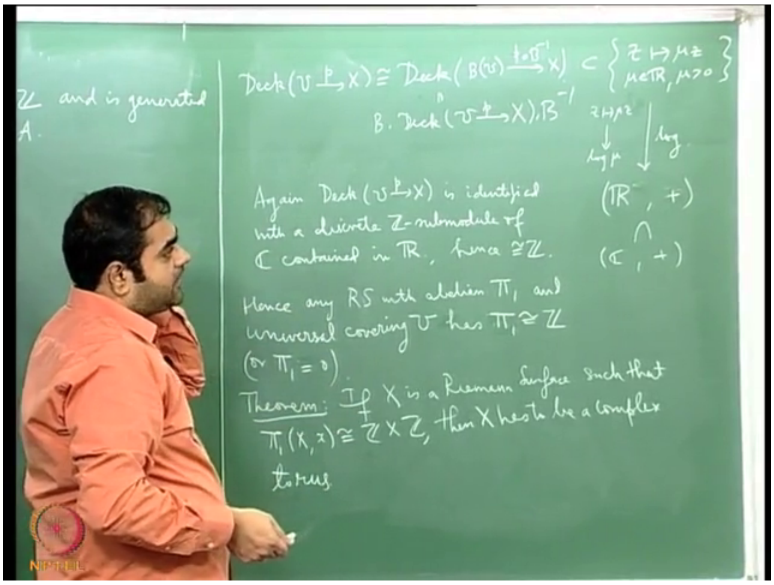

So, you see again what will happen is that, I will have deck transformation group of the original cover U p X isomorphic to the conjugate, which is deck transformation group of this isomorphic cover p sub to B inverse, which is just a conjugate of this by B it. So, it is B dot deck U p X B dot B inverse it is a conjugate, and well you see what I will what this is, this is subgroup of Mobius transformations of the form z going to mu z mu real mu positive ok. It is a subgroup of that. And you see notice that the composition here will be corresponding to composition here, but then it will become multiplicative. If I compose is z going to mu 1 z with z going to mu 2 z, I will get z going to mu 1 into mu 2 z. So, you know if I want identify it with additive sub module of c, I have to take a log and the log will convert this multiplicative group into an additive group alright and I can take log because mu is are all real positive alright.

So, what I do is, I simply take the log map; I take the log map namely I just sent z going to mu z to just log mu. So, this is the real logarithm to the base e right. So, I take this and what I end up with is well this is identified with this is a. So, what I get is going to be this is going to be identified with R plus under addition. So, R plus is why do I get oh I need also negative elements. So, I get R under addition right. So, correct I can have negative logarithms if mu is less than 1 correct. So, this identified as a additive subgroup of R. So, under this identification, the deck transformation group becomes an additive subgroup of R. Again it will become a discrete z module; it will again become a discrete z sub module of c the additive subgroup of c which is contained inside R.

So, again the same argument as before applies and it will tell you that it is isomorphic to zee. So, mind you this is contained in C comma plus; after all R comma plus is Z sub module of C comma plus alright. So, again deck U p X is identified with a discrete Z sub module of C contained in R. The discreteness can again be proved in a similar way by making use of these admissible neighborhoods. So, you again prove the discreteness alright and it becomes a discrete Z model of contained in R and therefore, it is isomorphic to zee ok.

Hence isomorphic to zee; so the moral the story is, if you have a Riemann surface x with up with fundamental group Abelian that is very important and universal cover the upper half plane, then it is fundamental group is just isomorphic to zee you do not have any choice alright. So, that is the upshot of the whole argument. So, hence any Riemann surface with Abelian pi 1, and universal covering U has pi 1 isomorphic to zee or pi 1 isomorphic to 0 or pi 1 equal to 0. Now well actually this argument gives you also the coverings of I mean also the classification of Riemann surfaces, which have upper half plane as universal cover, we have just have to go a little bit more and write it down, but what is more striking is that one can write down the following theorem.

So, theorem if x is Riemann surfaces such that pi 1 of x is isomorphic to zee cross zee, then x has to be a complex torus. So, here is an amazing theorem you take a Riemann surface with fundamental group isomorphic to zee cross zee then it has to be only a complex torus and what is the proof well it the fundamental group is not zero. So, the universal cover cannot be p 1 cannot be external complex plane C union infinity which is p 1, because in that case the fundamental group is 0.

If of course, the universal covering is the complex plane we have already proved that if the fundamental group isomorphic into zee cross zee, then it is a complex torus alright and the covering space cannot be U, the universal covering cannot be U because we have just now proved that if the covering universal recovering is U, then the fundamental group is isomorphic to zee and zee is not isomorphic to zee cross zee alright therefore, you see it is an amazing theorem, that you put a topological condition on the definition of Riemann surface and you it is the theorem tells you that it has to be of a particular type. So, you get that as of short of this right. So, let me make a few more remarks to complete this discussion.

So, let us look at both of these cases when the universal covering is upper half plane and either the deck transformation group contains only parabolic elements or only hyperbolic elements, and look at what the coverings are. So, let me write that down.

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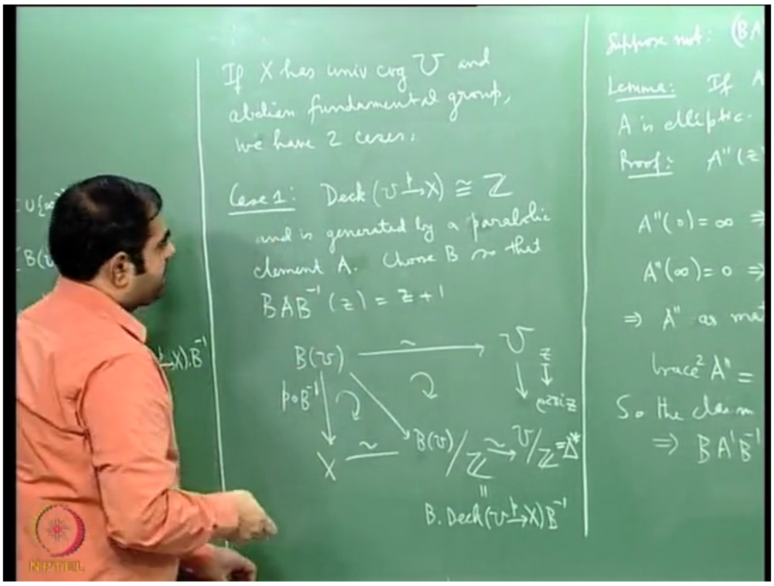

If x has universal covering U and deck the deck transformation group; so you should say and fundamental of be a and Abelian fundamental group, we have two cases. So, case one the deck transformation group is isomorphic to zee and is generated by a translation let me say by a parabolic element we have this case. We may choose; so let me take the generators as A choose B so that you see B A B inverse of z is z plus 1 ok.

So, in both cases you are conjugating the whole deck transformation group by a single Mobius transformation, that makes it into completely a group of translations in the parabolic case and in other case of course, it is say it becomes a multiplicative group of transformations of the form z going to mu z alright. So, we are in the first case right. So, you can be. So, without loss of generality choose the b, so that the translation that it is transformed to e z going to e plus 1 alright. Now see what will happen is. So, you see I have this diagram. So, I have B of U to X, I have this covering p circle B inverse and this is also the same as taking B of U and well going modulo the this translation group. So, this is going modulo zee.

So, you see zee is just identified with the conjugate of the deck transformation group by B, which is actually the deck transformation group of this cover alright and you see this covering map in this case it can be identified with. So, this is something that is isomorphic to U alright. So, it can be identified with U to U mod zee, it can be identified in this way alright. So, let me put it like this, you can find an identification like this and this map is just the exponential map, z going to e power 2 pi i z. So, what you must understand is that going we have already seen that the exponential map is a covering map from C to C star.

And, but if you instead of c star if you take delta star, which is the punctured unit disk the unit disk minus the origin and you take the inverse image, it is a restriction of a covering of a covering map to an open subset of the base so that continues to be a covering map. So, this is still a recovering map and the fact is that this is delta star; delta star is just the punctured unit disk, it is unit disk minus the origin. So, the moral of the story, you can check this the moral of the story is that in the parabolic case the recovering mind you, we already know that these two can be identified. If you give me a universal covering then the covering space modulo the deck transformation group can be identified with the base alright.

And I am saying that this can be identified as a covering with U to delta star the exponential map. Therefore, the moral story is that in the parabolic case your Riemann surface is actually bi holomorphic to delta star, and the covering map can be identified after choosing isomorphisms, you can identify it with the exponential map restricted to delta star. So, you get this this is the parabolic case alright ok.

Then let us go to. So, let me put this, the other case is the hyperbolic case that is when the deck transformation group consists of only hyperbolic elements. So, in that case again one can do the following computation.

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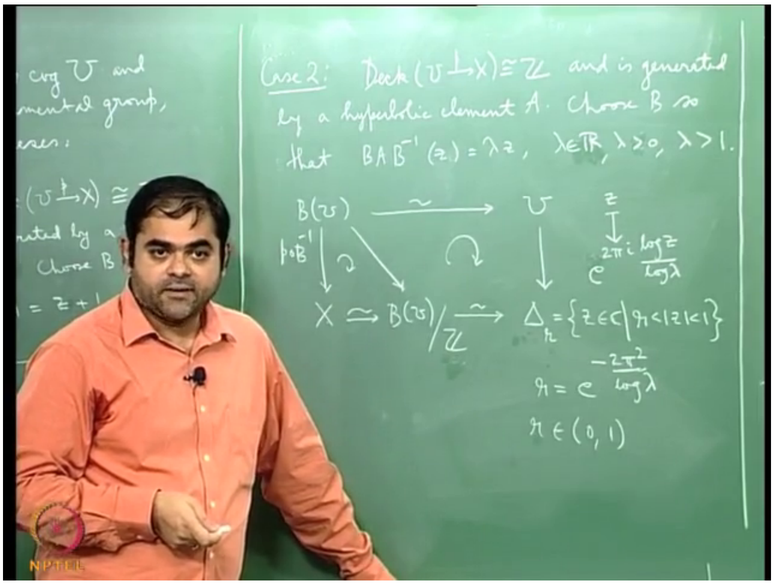

Case 2 the deck transformation group is a of course, isomorphic to z again and is generated by a hyperbolic element A. In this case you know of course, you know you can choose B A B inverse. So, that the element that B A B inverse you can choose B. So, the B A B inverses of form z going to lambda z lambda real lambda positive, and you know. In fact, I can make sure that the lambda that I get is less than 1. Because I have to only use an inversion and if I modify it by an inversion, if I conjugate by an inversion I can make sure the lambda is less than 1. So, let me write that down choose B so that B A B inverse of z is lambda z, lambda positive, lambda real of course, lambda real, lambda positive lambda. So, I need I can make lambda less than 1 or greater than 1. So, let me see what I want I want lambda greater than 1 ok, we can do this.

If lambda is less than 1 all you have to do is just an inversion, you have to conjugate further by an inversion. Then that will amount to replacing lambda by 1 by lambda which will be greater than 1 if lambda is less than 1. So, you can do this ok then. So, in this case what happens is that you see B of U. So, you have this covering p circle B inverse to X and this is of course, identified with B of U mod z, now this z being identified with this z here being identified with the conjugate of the deck transformation group as before.

So, I will it write it down, but I want to say is that I can identify this with again with U and I can identify this with delta sub r. So, this is something that I want you to check. So, this delta sub r is the set of all z in c such that, r is less than mod z less than one. So, it is an open annulus with outer radius 1 inner radius r, r is a fraction r is a r is a number between 0 and 1 alright and how is r given? R is given by a formula.

So, let me write down that formula it is e power minus 2 pi squared is e power minus of 2 pi squared by log lambda alright here of course, this is the this is the natural logarithm alright this is the value of r, and this map from U to delta r is none other than the map e z going to it is also an exponential map, it is e z going to e power 2 pi i log z by log lambda. I want you to check that this is a covering map alright which is quite easy to do if you know the properties of the exponential mapping; and mind you the logarithm that I am using here is the principle branch of the logarithm. When you say logarithm of a complex number you know it is a multi valued function.

So, you have to choose a branch and I am choosing the principal branch of the logarithm here and of course, this log lambda is of course, a real logarithm there is no problem with that and you can check that this is a covering map. So, the upshot of the story is that if you have a Riemann surface with universal covering U, either it is U itself is the same as delta or if it is this case it is delta star or if it is this case it is delta r, delta sub r for r a fraction lying in 0 1 ok

And. In fact, it is further also true that if I change r, then these delta r is not bi holomorphic to one another. So, that is something that you can check as an exercise, you by using universal properties by using the properties of universal covers you can check that. So, all these deltas rs are distinct for distinct rs. So, this gives you all possible cases for Riemann surfaces with Abelian pi 1 and universal covering u. So, what I want to say at this stage is that, we have kind of completed the first classification of all which includes all Riemann surfaces which have Abelian fundamental groups and we have also. So, what is left out is Riemann surfaces is non Abelian fundamental groups, and the compact Riemann surfaces of genus greater than 1, that is those complex structures Riemann surface structures that you can put on g torus g greater than 1.

So, these the g tori which have been stuck together, and you get a g torus and if g is greater than 1 then the universal cover has to be U, the fundamental group is not Abelian. Because of the fundamental group is Abelian we know all the cases, the fundamental group is non Abelian you can see it otherwise also, the universal covering has to be u. So, if you are interested in compact Riemann surfaces of higher genus, genus g greater than 1, then you will have to study the Mobius transformations on the upper half plane; that is the clue to studying compact Riemann surfaces of genus g greater than 1 that is what the study is.

So, I will end with that.