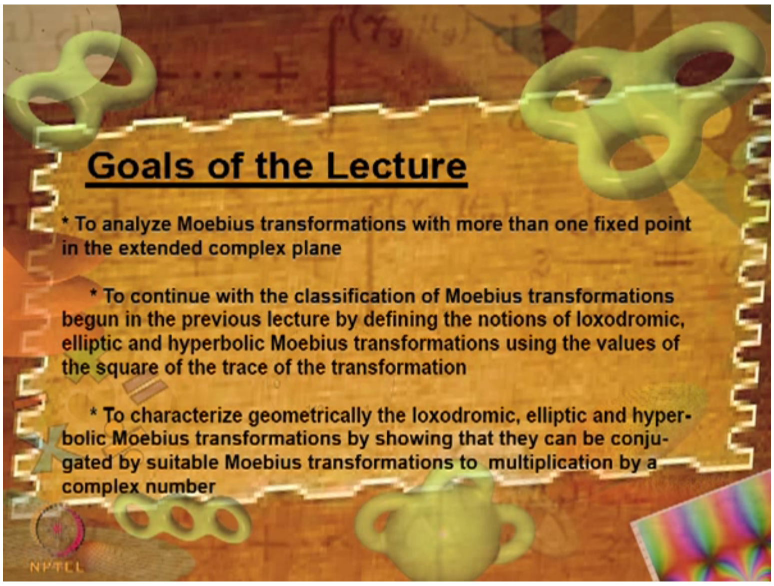
An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1 -dimensional Tori and Elliptic Curves

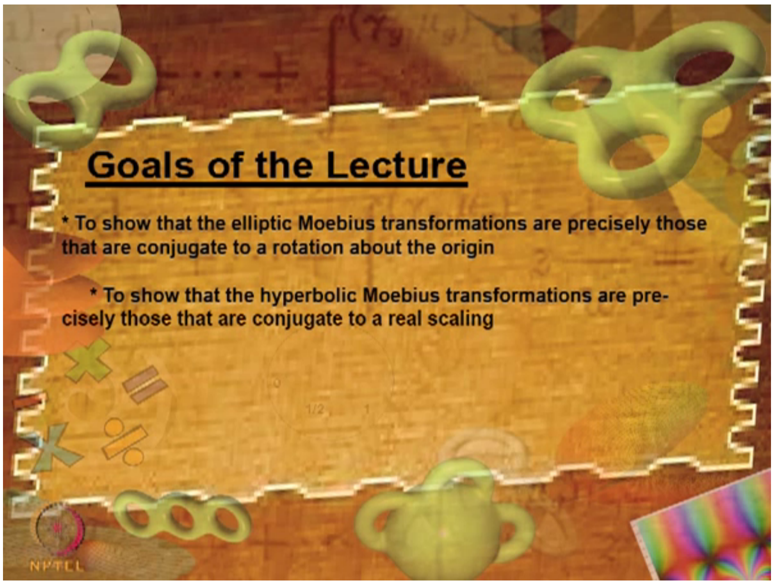
Dr. Thiruvalloor Eesanaipaadi Venkata Balaji
Department of Mathematics
Indian Institute of Technology, Madras

Lecture - 20
Characterizing Moebius Transformations with Two Fixed Points

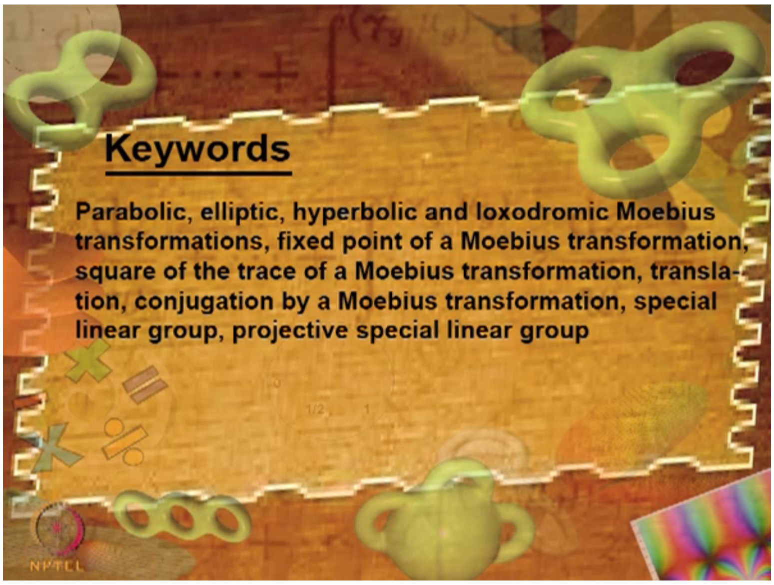
(Refer Slide Time: 00:10)



(Refer Slide Time: 00:16)

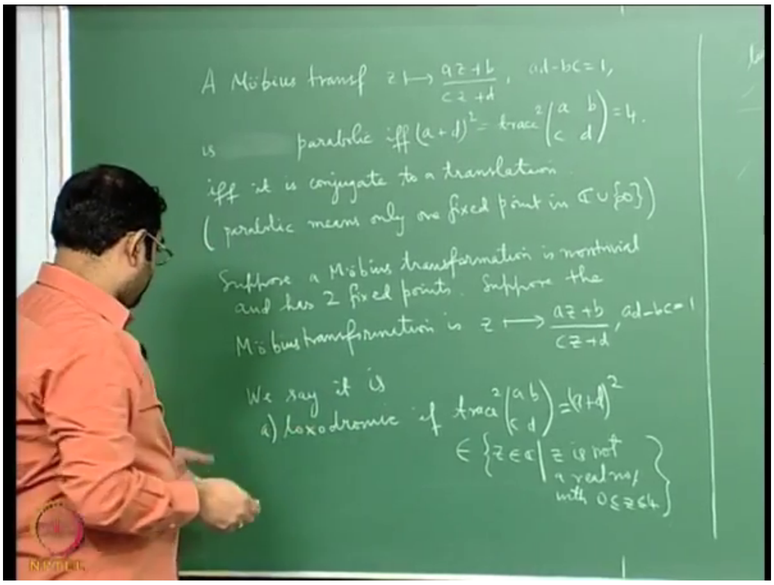


(Refer Slide Time: 00:21)



Let us continue with our discussion about Moebius Transformations.

(Refer Slide Time: 00:41)



So, we see yesterday, I was trying to tell you are rather in the last lecture, I was trying to tell you that a Moebius transformation is that going to a z plus b by c z plus d with the condition a d minus b c equal to 1 is called parabolic; parabolic if a plus d a plus d the whole squared which is actually trace squared of the matrix a b c d is the trace squared is equal to 4. So, this matrix a b c d is a representative of this Moebius transformation in sl 2 of course, the other representative is minus a minus b minus c minus d. And if you take the trace it will the trace squared it will be the condition that it is 4 is will give you the condition that it is parabolic. In fact, I what I proved yesterday I mean the last lecture was its parabolic if and only this is equal to 4 if and only it is conjugate to a translation. So, this is what I proved yesterday.

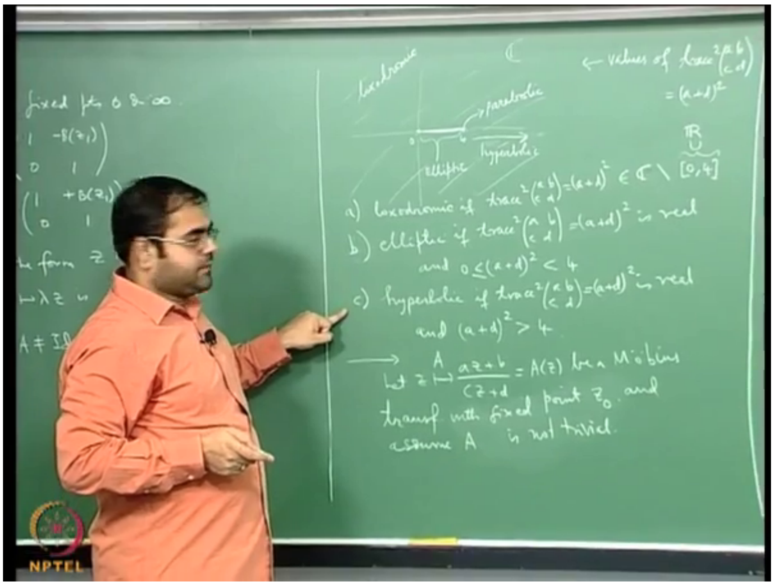
So, in particular and of course, let me recall. So, may; I should; maybe I should modify this should not say its parabolic is parabolic is parabolic if and only if this condition holds and of course, parabolic means only one fixed point in in c union infinity. So, Moebius transformation you know if it is not the identity, if it is identity it fix every point, if it is not the identity it can it will have at least on fixed point and maximum it can have 2 fixed points. And the case when it has only one fixed point is the case which is called parabolic all right and the point I wanted to say is that this parabolicity condition translates to condition on the trace squared that trace square has to be 4.

So, you can see that the way this is going the classification of Moebius transformation is being done in 2 ways by thinking along 2 directions one is by looking at the number of fixed points the other thing is by looking at trace square. So, let me define; let me look at the other cases let me look at the other cases. So, suppose; suppose a Moebius transformation is nontrivial; it is not the identity transformation and has 2 fixed points. So, the case when it is one fixed point is called parabolic the case when. So, what is left out is the case when there are 2 fixed points. So, this is the non-parabolic case.

Now, what does how do we classify Moebius transformations which have 2 fixed points. So, the key to is; this is again the looking at the values of trace squared. So, suppose the Moebius transformation the Moebius transformation is is given by z going to a z plus b by c z plus d as usual with a d minus b c equal to 1. So, again the point is to look at the values of trace squared of the representing matrix in s l 2. So, you see we say; we say; it is a loxodromic if the trace squared a b c d which is a plus b the whole square does it does not.

So, this is the complex number mind you a b c d are complex numbers all right and in general when I calculate the a plus b the whole square I get a complex number what I want it to be I want it to be a complex number which is not a real number line from 0 to 4. So, this belongs to set of all e z in c such that is that. So, I should say e z is e z naught a real number in with 0 is not equal to z less than and equal to 4. So, this is the condition,.

(Refer Slide Time: 07:25)



So, you know if I try to draw a diagram I have the whole complex plane and then I have this line segment here from 0 to 4; I have this line segment all right if. So, this is the line segment here and if your trace if you calculate trace squared and does not that value does not lie on this line segment that is it lies outside the compliment of this line segment then it is called loxodromic; it is called loxodromic all right notice that if trace squared. So, what is left out is the case when trace squared is real and lies between 0 to 4 that is what it left out. So, that gives you 2 other cases.

So, let me write this down its called. So, let me first say elliptic if trace squared which is a plus d the whole squared is real and it lies between 0 and it strictly less than 4 of course, you know trace squared equal to 4 is parabolic. So, this point is parabolic. So, I am looking at here I am looking at values of trace square which is a plus d the whole square. So, when it is 4 its parabolic if it is from 0 to four, but not equal to 4, then it is called elliptic and of course, if it is real and its value is greater than 4 then it is of course, loxodromic, but its given a given a special name it is called hyperbolic. So, hyperbolic if trace square a b c d is a plus d the whole square is real and a plus d the whole square id greater than 4. So, you see this this region its elliptic not inclusive of the point four, but everything from 0 up to something less than 4 up to all value less than 4 its elliptic and then from beyond 4 onwards. So, from here onwards it is elliptic I mean it is hyperbolic. So, this is the; these are the definitions.

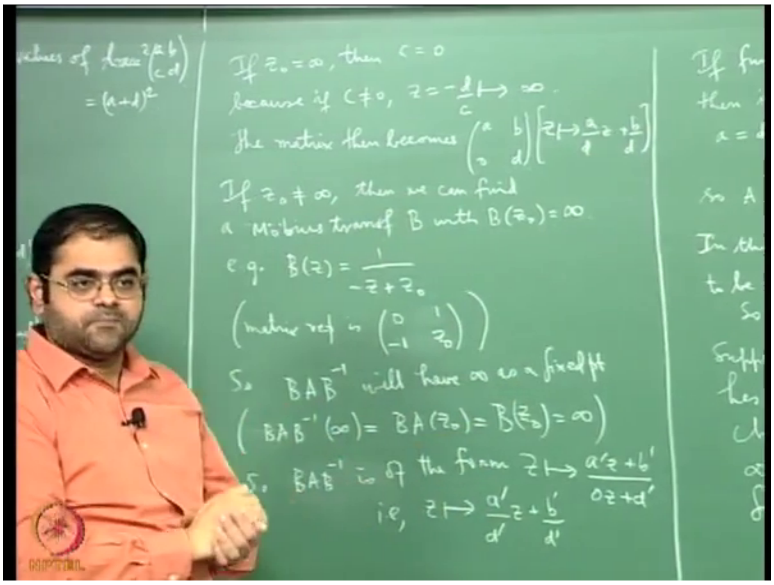
And you can see that all these definitions are I mean the definitions are mutually exclusive except for the fact that hyperbolic transformation is a very special type of loxodromic transformation and. So, you have elliptic. So, if trace squared is real then trace squared has to lye if it is lies between 0 and 4 then it is elliptic if it is equal to 4th parabolic if its greater than 4 its hyperbolic and any other value if it takes then it is loxodromic. So, these are the important definitions. So, the points that these definitions come from looking at this quantity namely the trace squared of a matrix representative of your Moebius transformation in s l 2.

So, what do you must understand is this this this quantity the trace squared seems to do the job of you know trying to classify by them. So, let us study in detail what these cases are. So, well I already told you what parabolic transformations are they are the once which have only one fixed point and they are essentially translations, but you know Moebius general Moebius transformation consist of various thing I mean you can break it down into translations rotations and scalings and of course, you can also break it down into inversions. So, you can factor it out.

So, we have the parabolic case corresponds to the translations all right the other cases come here. So, let us let us right analyze this. So, let me look at let me look at a Moebius transformation of this type. So, let e z going to a z plus b by c e z plus d let me call this as a. So, this is a of e z be a Moebius transformation with 2 fixed points let us take a Moebius transformation with 2 fixed points or let me even begin by taking with it will have if you assume that this is not the idea in transformation; it will have at least one fixed point. So, let me first take that case let this be a Moebius transformation with fixed point e z naught let us say. So, and assume and assume a of z is not a of z is not there is not the identity transformation. So, rather let me see a is not trivial, a is not trivial means a is not the identity transformation right. So, it has at least one fixed point and it could have 2 fixed points.

Now, you see the first thing I want to claim is that if e z naught is infinity, then c has to be 0. So, let me write that down.

(Refer Slide Time: 15:01)



If e z naught is equal to infinity then a claim that c is equal to 0, why is that true because if c is not equal to 0 then I can make the denominator vanish by putting z equal to minus d by c; ok and if I put z equal to minus d by c the denominator vanishes therefore, this quantity will become infinity. So, that is how the point at infinity is also included in the with along with the operations on the complex numbers; if you remember any finite complex number divided by 0; it is taken as infinity. So, if I put z equal to minus d by c, then minus d by c is a finite complex number; it is not infinity and it goes to infinity.

So, this goes to this goes to infinity now, but you know all your Moebius transformations are bijective and by holomorphic maps of the extended complex plane onto itself c union infinity onto itself therefore, if minus d by c a finite complex number goes to infinity, then infinity is not a fixed point because of injectivity. So, c has to be 0. So, what it means is that the matrix the matrix form the matrix then becomes you know a b 0 d and. In fact, these are precisely the Moebius transformations which are auto morphisms of the complex plane those though what are the Moebius transformations which are auto morphisms of the complex plane they are all those Moebius transformations which keep to point at the infinity fixed and for keeping the point at infinity fixed this is the upper triangular matrix form that you will get.

So, this Moebius transformations of this type they form auto morphisms the holomorphic auto morphisms of c and we use to denote that as p delta 2 comma c; so fine. Now, suppose. So, you know if z naught is infinity, then it is already in that form, but suppose z naught is not equal to is not the point at infinity suppose z naught is not the point at infinity. So, here by the way I will make that remark later. So, if z naught is not the point at infinity, if z naught is not the point at infinity, then we can find a Moebius transformation b with b of z naught is equal to infinity, you can do this you can move z naught infinity you can move z naught infinity; for example, you could have taken b of z to be simply 1 by z minus z naught if you take b offset is equal to 1 by z minus z naught then z naught will go to infinity. So, it is a Moebius transformation that we move z naught to infinity.

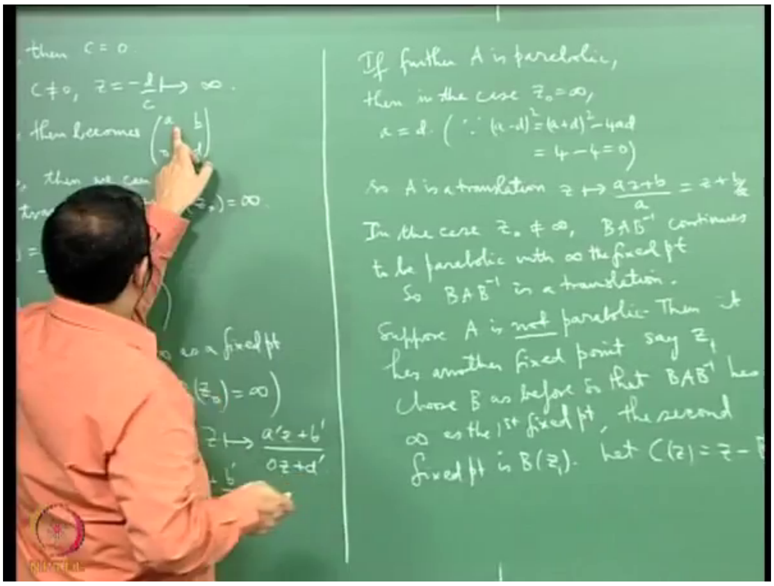
And you see mind you the matrix representation is you know 0 1 1 minus z naught and if you want a matrix representative in s l 2 c, I want determinant to be 1. So, what you should do is you should you can change sign everywhere. So, you I put minus minus here and I will put plus here. So, this is a matrix of representative does that help; probably, now it is still minus 1. So, maybe I will have to take z naught minus n right, I will have to take minus z plus z naught if I take minus z plus z naught, then it should work. So, it is going to be 0 1 minus 1 z. This is now in s l 2, right; so just to keep track of the matrix substitute right, fine. So, the point is why I am doing that is because you know if you want a matrix representative you must make sure that it is a representative in s l 2. So, you must adjust it. So, that the determinant is 1 ok.

So, if I put z equal to z naught b of z naught is going to be infinity now watch if I take BAB inverse if I take this BAB inverse if I take this Moebius transformation namely I conjugate A by the Moebius transformation B,, then this Moebius transformation will have infinity as a fixed point will have as a fixed point. So, you will get that is and that is because you know BAB inverse of infinity B takes z naught infinity. So, b inverse way of infinity will be z naught. So, I will get BA of z naught, but z naught is a fixed point of A. So, A z naught is a is just z naught. So, this is just be z naught and B z naught is like actually infinity. So, you see BAB inverse of infinity is infinity moral of the story is if you have a fixed set point at z naught I can always conjugate it.

So, that the new transformation after conjugation has fixed point at infinity. So, this means that if I write BAB inverse its matrix of if I write it out then the matrix form has to be it should look like this. So, you see BA matrix representative so. In fact, let me write this. So, BAB inverse is of the form is of the form e z going to let me use a prime e z plus b prime by 0 z plus d prime because I already told that whenever the for a Moebius transformation if e z is a fixed point then this member this entry has to vanish. So, it will look like this.

Now you see. So, that is this is what this is just z going to a prime by d prime e z plus b prime by d prime and notice that d; d is not 0 because then you know ad minus b c is one and since c is 0 ad is 1. So d is not 0, mind you. So, dividing by d here or dividing by d prime here is allowed there is no problem there. So, now you see that you know if you put further the condition that A has only one fixed point suppose I put the further condition that a has only one fixed point; that means, a is parabolic then I claim a prime is equal to d prime and it is a translation ok.

(Refer Slide Time: 23:46)



If further a is parabolic, then in the case z naught equal to infinity which is here a must be equal to d and you know why this is true because you know since as I explained in the last lecture a minus d the whole squared is a plus d the whole squared minus 4 a d all right and well a plus b if it is parabolic a plus b the whole squared is 4 and a d a times ad minus bc is is going to be 1, but c is 0. So, ad is one. So, this is. So, you see this is going to be 4 minus four. So, it is going to be 0. So, a will be equal to d and so, a is a translation it is a translation namely e z going to a z plus b by d which is just the same as e z plus b by a. So, actually in this case you can see it you do not have to do any conjugation.

All right, if a is parabolic and infinity is the only fixed point, then it is already a translation all right if in the case. So, in the case when e z naught is not equal to infinity then you choose b like this we choose be like this BAB inverse continues to be parabolic with infinity a fixed point infinity the fixed point because you see the property of a Moebius transformation being parabolic or elliptic or loxodromic or hyperbolics is not going to change when you conjugate it that is because that property is defined by looking at the value of trace squared of this matrix and you know the trace of a matrix is invariant under conjugation that is you take a matrix you pre multiply it by an invertible matrix and post multiply it by its inverse the inverse of the pre-multiplied matrix then the trace continues to be the same. So, if I take any of any Moebius transformation with any of these properties and if I conjugate it I will continue to get a Moebius transformation with the same properties, if I conjugate a loxodromic transformation, I will again get a loxodromic transformation if I conjugate an elliptic transformation, I will again get an elliptic transformation and so on.

So, since I have assumed A is parabolic BAB inverse continues to be parabolic and infinity is a fixed point. So, you are in this case. So, BAB inverse is a translation. So, this gives you the proof of this another proof of this statement that I was giving you in the last lecture that you know for a parabolic for transformation to be parabolic it should be conjugate to a translation. So, the last time I gave you a; I was looking at a different proof because I wanted to show to you that if you try to solve for b. So that be a b inverse is a translation say even translation by 1 it poses the condition that the trace square should be 4.

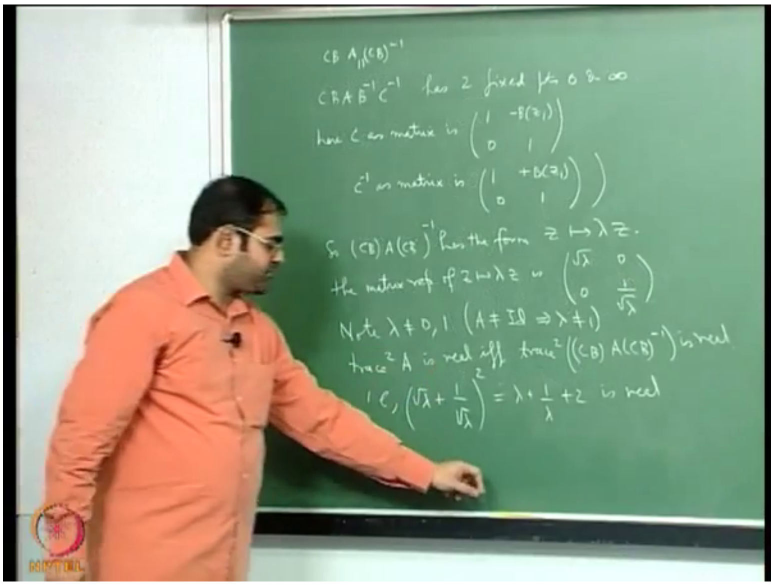
So, this is another direct way of seeing that right now. So, this is the this is a situation when there is only one fixed point, but let us assume that there is another fixed point suppose a is not parabolic. So, then it will have another fixed point, then it has another fixed point say e z one. So, e z 1 is another fixed point for a so; that means, you know we are now in these remaining cases. So, a has to be either; it has to be loxodromic or elliptic or hyperbolic and we want to see what kind of form a has.

So, you see if it has another fixed point say e z 1 you will see that well in either of these cases I can always find a b such that infinity is a fixed is the first fixed point. So, again choose b as before. So, that BAB inverse has infinity has the first fixed point first fixed point that I can do and mainly of course, if I mean in this case I have to choose b to be identity because I do not have to conjugate a talk I do not have to conjugate if I am in this case if already the first fixed point is infinity I do not have to conjugate; I can take b equal to identity.

If the first fixed point is not infinity then I have to conjugate. So, that the first fixed point becomes infinity. So, if I take BAB inverse infinity will be the first fixed point, but now the second fixed point is changed to b of z 1 the second fixed point fixed point this b of z 1 that is because you see if I calculate BAB inverse of b of z 1 I will get BA of z 1 which is b of z one. So, I will get that this is a fixed point that is obvious all right now what I am see I have already moved by a conjugation I have moved one fixed point to the point at infinity and you know what I am going to do you can you can guess this I am trying to the other fixed point is not the point at infinity it is a different fixed point and you know I will again do some do a conjugation move it to 0.

So, I try to bring a to a form of a Moebius transformation which has 2 fixed points namely 0 and infinity. So, that is what I am going to do. So, for that to make this fixed point to go to 0 I just have to translate by minus of this. So, let see the let us see of z they just z minus b of z 1 z minus b of z 1. So, this is translation by minus be of z 1.

(Refer Slide Time: 31:46)



So, what happens is if I take CB; AB inverse, C inverse, if I take this Moebius transformation all right which is which is incidentally also the same as CB A CB inverse if you want these are the same CB inverse is B inverse C inverse.

If I take this Moebius transformation then you will see its 2 fixed points are going to be 0 and infinity mind you c here c as matrix is you know its translation by minus z minus b of z 1. So, it is going to be one minus B of z 1 0 one which is already in s l 2 this is a translation by minus B of z 1 and what is c inverse the inverse of a translation is just translation by the negative of the vector that you originally translated with. So, c inverse as a matrix is just going to be one plus b of z 1 of 0 one this is what seen what is going to be the inverse of translation is just another translation right.

So, if I take this then I will get 2 fixed point 0 at infinity. So, you see CB ACB inverse has the form is it going to lambda z if infinity you see if infinity is a fixed point already cis 0 and it is and it becomes the your Moebius transformation becomes A by D times z plus b by D e z going to A by D a by d e z plus b by d all right that is what this means e z going to a by d z that is B by D it comes to this form all right if infinity is a fixed point.

And now if I say that 0 is also a fixed point this this constant term cannot come because 0 has to go to 0; that means, b has to b 0 and if b is 0 then it is just going to be e z into some complex number and that complex number I you can one can go I am calling it as lambda. So, the moral of the story is you take a Moebius transformation that is not parabolic then you can always do a conjugation and bring it to this form now the question is you can you can guess what I am trying to get at I am trying to see what properties of lambda will make it you know loxodromic or elliptic or hyperbolic that is the point and it turns out that nice prop there are very simple conditions on lambda that can tell you whether this is you know one of those types.

So, let us look at those situations. So, you know the matrix representative of is that going to lambda z is well you know it will be square root lambda 0 0 1 by square root lambda see if I write lambda e z just as a matrix I will simply get lambda 0 0 one if I write lambda if I write e z going to Az plus B by Cz plus d as a matrix a b c d then e z going to lambda z the matrix if I write as a matrix it will be just lambda 0 0 one, but the determinant of that matrix is lambda, it is not an sl 2 c representative because I i am look I am only looking at sl 2 c representatives for my calculations especially when I am when I do this trace business. So, the 2 make it an element of sl 2 ci take a representative like this of course, I can also put minus I can take minus root lambda minus 1 by root lambda and mind you root lambda is 1 square root of lambda there are 2 in general there are 2 square roots of lambda and I take anyone square root and use the same square root here in both cases.

And mind you note that lambda is not 0 its not one because you know if lambda is 0, then that is not a Moebius transformation it is just the 0 map, it is not a Moebius transfer and if lambda is 1 then this is identity if CB ACB inverse is the identity. So, I think the inverse has been wrongly put it should be put outside if you conjugate a matrix and you get identity then that matrix itself has to be identity, because you can unconjugated and on the right side because you have identity it will continue to be identity.

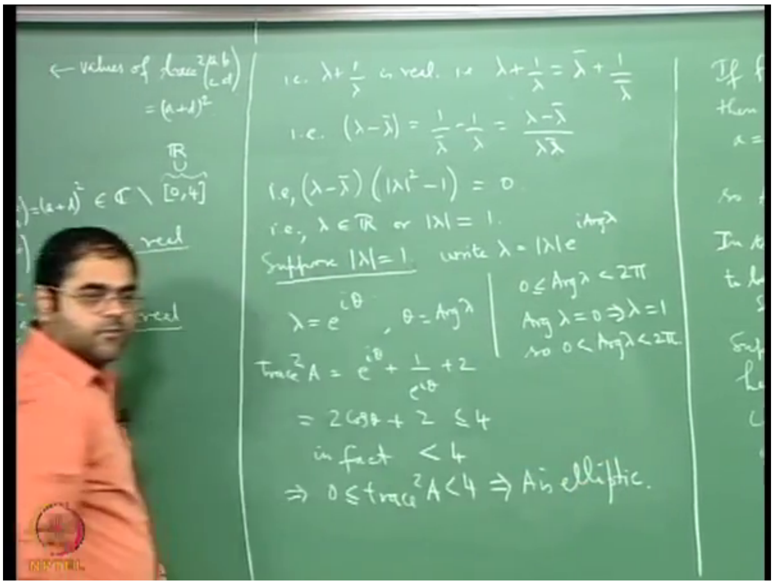
So, this is not being the identity will tell you that lambda is not equal to 1 a not equal to identity implies lambda not equal to 1. So, you see the lambda value is lambda equal to 0 and 1 are forbidden lambda can only have values other than 0 and 1. So, you see when I gave this definition of. So, I have rubbed it the loxodromic condition. So, let me write it here let me rewrite it here loxodromic if trace square a b c d which is a plus d the whole square is in c is a complex number not a real number in this interval. So, this see I am considering this as subset of real numbers ok.

So, if the trace squared does not take a value on this segment, then that is my definition of loxodromic all right of course, trace squared is not 4 because there are 2 fixed points and it is not parabolic. So, in all these let us look at these 2 cases you see in these 2 cases you see the trace squared is real and the loxodromic case the loxodromic case with which is not hyperbolic, because hyperbolic also implies loxodromic the loxodromic non hyperbolic cases the case when your trace is that is a condition when the trace does not lie in the segment. So, what I am going to the; I am going to first look at the condition that trace squared is real.

So, you see trace squared a is real if and only if you know trace squared is the same as trace squared of this trace squared of for CB ACB inverse is real, but CB ACB inverse the matrix is this and what is the trace its root lambda plus 1 by root lambda that is a trace and this trace squared will be root lambda plus 1 by root lambda the whole square. So, that is root lambda plus 1 by root lambda the whole squared is real this is what you will get I mean this is the condition you will put if you want to study something that is elliptic or hyperbolic all right so, but what is this well this is just I expanded this lambda plus 1 by lambda plus 2 this is real ok.

So, well let us investigate the situation. So, let me rub this off. So, I will keep I will keep this part I will keep this part as it is all right and I will continue here because I would like to you to keep these definition in mind. So, you see. So, of course, lambda plus 1 by lambda plus 2 is real if and only if lambda plus 1 by lambda is real ok.

(Refer Slide Time: 41:14)



So, that is lambda plus 1 by lambda is real all right and what is the condition this this happens if lambda plus 1 by lambda is equal to lambda bar plus 1 by lambda bar you know a complex number is real if and only if it is equal to is conjugate and then I can collect terms I will get lambda minus lambda bar is equal to 1 by lambda bar minus 1 by lambda which is again lambda minus lambda bar by lambda lambda bar and you know lambda lambda bar is just mod lambda the whole squared if I cross multiply I will get lambda minus lambda bar into mod lambda the whole squared minus 1 is equal to 0, I will get this condition if I collect terms lambda lambda bar is mod lambda the whole squared.

So, when will trace square A plus D B real trace square A plus D will be real I mean a square of a which is a plus b the whole squared will be real if and only if this transformation that I have got gotten after conjugation which is essentially is that going to lambda z the lambda satisfies this property either lambda is equal to lambda bar which means lambda is real or mod lambda is 1. So, let me write that down. So, you get these you get these 2 conditions of lambda.

Let us investigate these conditions suppose mod lambda is 1 we will show that mod lambda is 1 is the case when it is elliptic why is that true well you see. So, you know lambda you write lambda as you know mod lambda e power I argument of lambda and well 0 is less than argument of lambda is less than 2 pi. So, you have the argument should vary over an interval of length 2 pi all right and of course, I exclude 2 pi and I include 0. Now you see notice that argument lambda e equal to 0 is not allowed because you know if argument lambda is equal to 0 all right then I will get lambda equal to mod lambda.

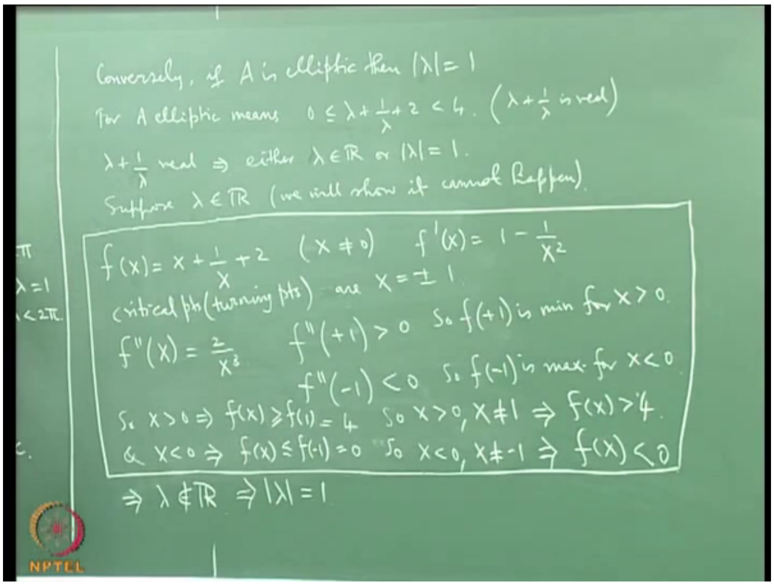
So, I will get lambda equal to 1, but lambda equal to 1 is not allowed. So, argument lambda is equal to 0 implies lambda equal to 1. So, you see 0 should be strictly less than argument of lambda strictly less than 2 pi. So, what happens? So, let me write this as. So, lambda is e power I theta where theta is argument of lambda. So, I am in this case and look at look at that quantity trace squared A is just e power I theta plus 1 by e power I theta plus 2.

So, it is going to be this is cost by formula this formula this is you know this is cos theta plus I sin theta is cos theta minus I sin theta. So, I will get 2 I will get 2 cos theta minus 2 plus 2 and you see this quantity you know cos theta is bounded by 1. So, the first thing I need to say. So, I just want to say that this is in any case this is less than or equal to 4, this is less than or equal to 4 and mind you. So, I want to make the statement that this quantity is going to lie between 0 and 4 it cannot be if equal to 4.

So, you see if this is equal to 4 then you know cos theta has to be one, but cos theta will not assume the value one so. In fact, less strictly less than 4 and I claim that this quantity is actually positive is greater than equal to 0 this quantity is greater than or equal to 0 because you know why is that so; that is because I think that is also quite obvious I mean you know cos theta at the minimum can become minus 1 and in that case, I will get 0 cos theta is always greater than minus 1 greater than or equal to minus 1 and I am adding 2, I am multiplying it by 2. So, 2 cos he has greater than equal to minus 2 and if I add 2 it is always greater than equal to 0. So, this implies that you know 0 less than or equal to trace squared a strictly less than 4 and I will tell you that a is elliptic that is the that is a condition and mind you what it will actually tell you is that this is elliptic, but then I told you by after conjugation the ellipticity does not change.

So, the original transformation that you started with is also elliptic. So, the moral of the story is you take a non parabolic transformation you conjugate it. So, that it becomes z equal to lambda z a z. So, it becomes z go going to lambda z and if mod lambda is 1 then it will be elliptic and the converse also holds if this is elliptic, then mod lambda has to be 1.

(Refer Slide Time: 48:38)



So, how does one see that? So, let me write that down conversely; if a is elliptic then mod lambda is 1 if a is elliptic I claim mod lambda is 1 why because you see because of the following reason for a elliptic means you see 0 less than or equal to lambda plus 1 by lambda plus 2 less than or let us strictly less than 4 all right that is what it means it means this is real lambda plus 1 by lambda is real. So, if it is elliptic what is the definition for elliptic a plus c the whole square is real and a plus c the whole square turns out to be this this is real. So, lambda plus 1 by lambda is real and its value lies between 0 and 4 0 included 4 excluded this is the condition for it being elliptic now from this you can conclude that mod lambda is 1 and how does one show that I think it is going obvious.

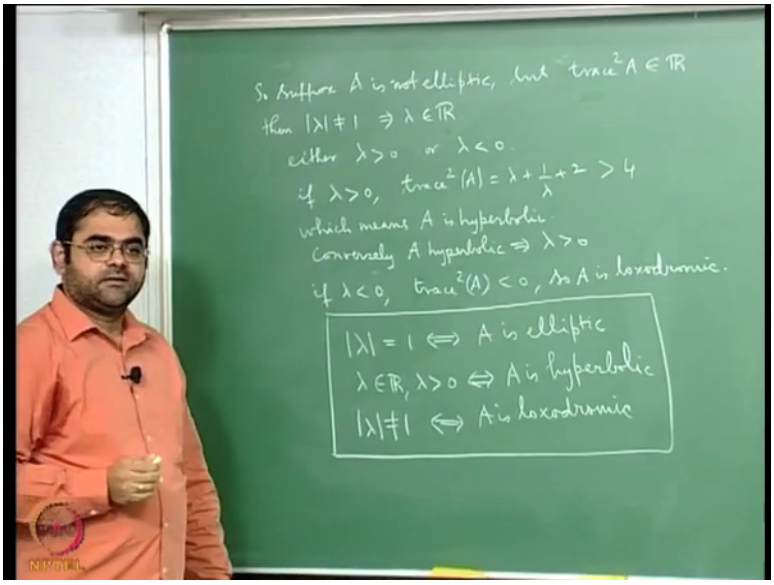
So, you know what will happen lambda plus 1 by lambda real would mean that you know either lambda is equal to lambda bar or mod lambda is 1 of course, we are trying to prove mod lambda is 1. So, what we will have to do is you have to exclude the case when lambda is real; suppose lambda is real. So, this is the case you have to study. So, we will show this it cannot happen we will show it cannot happen why because you see you take the function f of x is equal to x plus 1 by x plus 2 take this function then of course, x not equal to 0 then you will see that is derivative is you know one minus 1 by x square and therefore, critical points or turning points of the graph of the function are going to be are x equal to plus or minus 1 I have to equate this to 0.

And the second derivative or the function is going to be 2 by x cube the second derivative at plus 1 is positive. So, at plus 1 I have minimum. So, f of plus 1 is minimum for x positive and the f f if I take the second derivative at minus 1 then I get a negative value. So, I will get f of minus 1 is maximum for x negative and what are the values. So, you see. So, x positive implies f of x is always greater than f of 1 greater than or equal to f of 1 and f of 1 is 4 and so, x positive x not equal to 1 will imply f of x greater than 4 and x negative will imply f of x is less than or equal to f of minus 1 which is 0. So, x negative x not equal to minus 1 will imply that f of x is strictly less than 0. So, this is a very simple calculus computation, but it is very helpful to us.

So, you see if now you see if lambda is a is real then there are 2 cases lambda can be either positive or negative all right all right and well my situation is that. So, what is given here is that if lies between 0 and four. So, if I instead of x if I plug in lambda if I take f of lambda then f of lambda is either greater than 4 or it is less than 0 if lambda is real what is the upshot of this if x is real the expression x plus 1 by x plus 2 is either greater than 4 or it is less than 0 provided x is not allowed to take the values plus or minus 1; so that will give you a contradiction to this case that. So, lambda cannot be real. So, the lambda cannot be real and lambda cannot be real the other case will be that lambda has to be one mod lambda has to be 1. So, what we get is that the transformation is elliptic if and only if after conjugation to make it into the form that going to lambda z mod mod lambda is equal to 1.

So, mod lambda equal to 1 is a condition for it to be elliptic all right incidentally there is the other case when lambda itself is real that also has been taken care of here if you watch carefully. So, let me put this in a box I mean this is A is a rather simple computation, but it is very helpful. So, you see let me go back here and look at let me look at the other cases. So, suppose A is not elliptic, ok.

(Refer Slide Time: 56:06)



Suppose A is not elliptic, but you see lambda, but trace squared A is real. Suppose A is notelliptic and suppose trace squared a is real. So, then mod lambda is not equal to 1 because a is not elliptic because you have proved just now that mod lambda being one is exactly the case when a is elliptic. So, mod lambda not equal to 1 will imply that lambda is real you will get the case that lambda is real. Now you see if lambda is real you look at you look at this the situation.

So, I have to say something .So, there are 2 cases either lambda is positive or lambda is negative if lambda is positive all right then you know trace squared a is just lambda plus 1 by lambda plus 2 that is greater than 4 here if x is greater than 0 and x is not equal to 1 then x plus 1 by x plus 2 is greater than 4. So, trace squared a becomes greater than 4 and that is the condition for a to be hyperbolic which means A is hyperbolic conversely if you a is hyperbolic all right then I claim that the condition is lambda is greater than 0.

Conversely a hyperbolic implies lambda greater than 0; you get that. So, and why is that true because you see if a is hyperbolic all right then the condition is trace squared is real and trace squared is real will mean either lambda is or mod lambda is 1, but a is hyperbolic and. So, mod lambda cannot be one because then if mod lambda is 1 A is elliptic. So, lambda has to be real and if lambda has to be real and we have this condition hyperbolic static condition that lambda plus 1 by lambda plus 2 is greater than 4 then lambda has to be positive because if lambda is negative lambda plus 1 by lambda plus 2 is negative.

So, the condition that A is hyperbolic is controlled by this lambda being real and positive. So, is that going to lambda is that should have the property that lambda is real positive that is the condition for A to be hyperbolic. So, we get that. So, I i have this the other case lambda less than 0 lambda is negative we have seen that we have seen here that lambda plus 1 by lambda plus 2 is less than 0 and therefore, you are in the loxodromic case. So, if lambda is less than 0 trace squared A is negative. So, A is loxodromic. So, the upshot of this whole discussion is the following lambda mod lambda equal to 1 if and only if a is elliptic; lambda real lambda positive if and only if A is hyperbolic and mod lambda not equal to 1 if and only if a is loxodromic.

So, these 3 simple conditions will allow you to distinguish between elliptic hyperbolic and loxodromic transformations and somehow we have arrived at these conditions by start looking at the trace squared which was originally used to define these things.

So, we will continue in the next lecture.