An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-dimensional Tori and Elliptic Curves
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Lecture – 19
Characterizing Moebius Transformations with a Single Fixed Points

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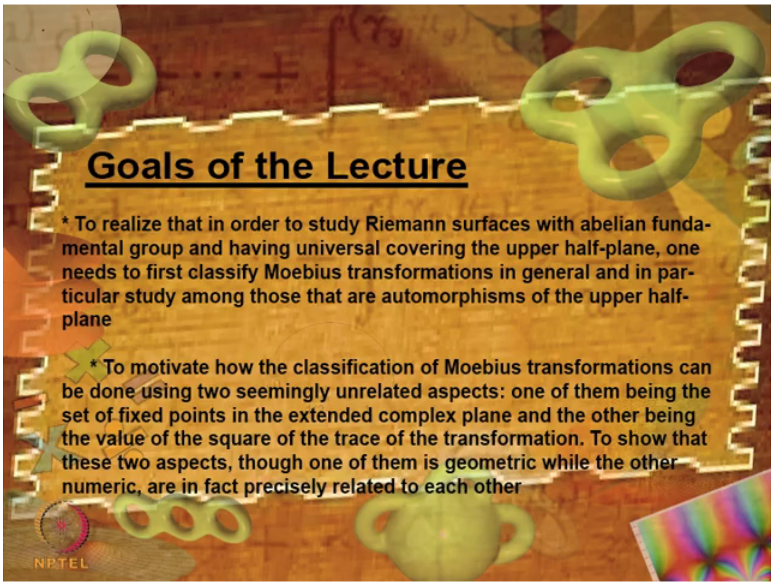

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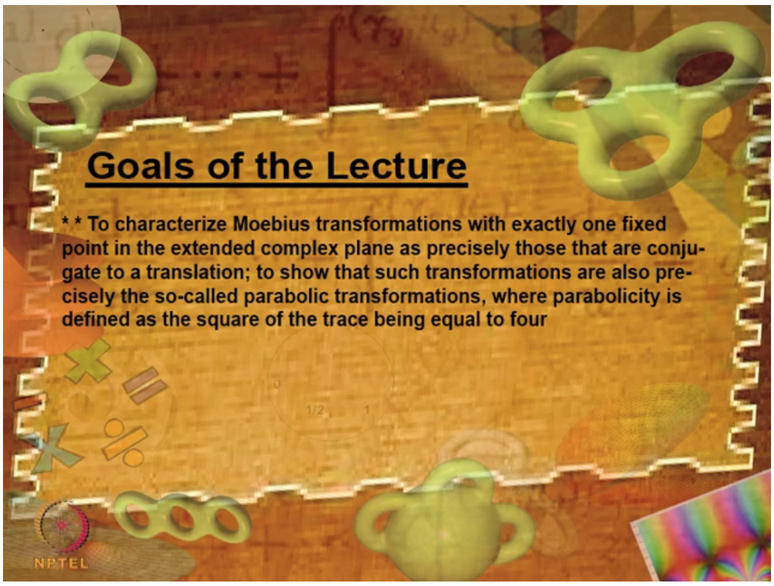

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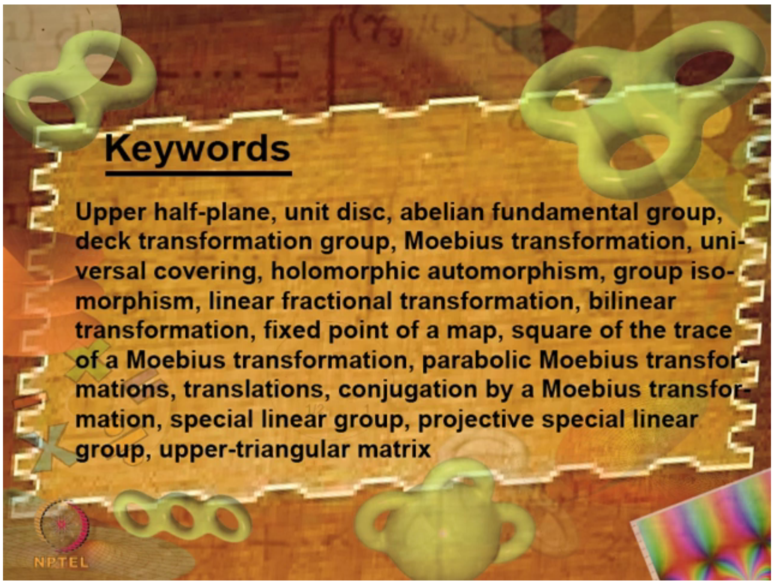

So, let us continue with the trying to classify Riemann surfaces. So, you see we have already looked at two cases when the universal covering of Riemann surface is p 1 Riemann's sphere then we have seen that the Riemann surface itself has to be p 1 that is holomorphically isomorphic to p 1 which is c union infinity via this stereographic projection. And you also seen that if a Riemann surface has universal cover the complex plane then it is fundamental group has to be Abelian and there are only three cases the fundamental group being 0 in which case it is by holomorphic to the complex plane then the fundamental group is being z isomorphic to z the integers under addition in which case it is you know Riemann surface structure on a cylinder on a real cylinder. And the standard covering map in this case is just the exponential map from c to c star the punctuate c star is a punctured complex plane and c to c star is a exponential map. So, this is the single a representative.

And when the fundamental group is isomorphic to z cross z then the Riemann surface is a complex stores. So, these were the cases when the universal covering was the complex plane.

Now, we have to look at those Riemann surfaces with universal covering the upper half plane which is the only case that is left and of course, instead of the upper half plane we can also choose the unit disc because they are by holomorphic to one another and I will only begin with trying to look at those Riemann surfaces which have universal covering the upper half plane and which are Abelian. So, in this way I would have covered classifying all Riemann surfaces which have Abelian fundamental groups, fine.

So, how does one proceed? So, one needs to somehow try to answer the following question what are the kinds of Moebius transformations that you can find in the deck transformation group. So, that is the question that needs to be answered right. So, let me write that down.

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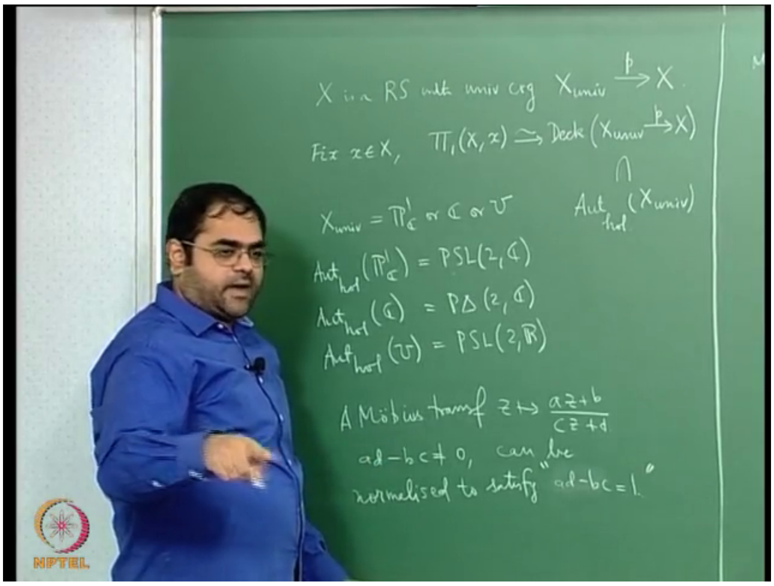

See you know that X is a Riemann surface with let me say universal covering X sub univ see you have you have this, this universal covering. Then you know that if I fix a point small x in capital X, then the first fundamental group of the base at based at the point small x is naturally identified with the deck transformation group of this universal cover which is a subgroup of holomorphic automorphisms of the universal cover.

So, of course, the universal cover modulo this deck transformation group is precisely X. So, the question is what are the elements, what are the elements here. See you know, this has only three possibilities you know X sub univ is a Riemann surface and it is a simply connected Riemann surface. So, this is either p 1 which is the Riemann's sphere or it is a complex plane or it is the upper half plane.

And each of these cases the group of holomorphic automorphisms are Moebius transformations. So, automorphisms the holomorphic automorphisms of p 1 are going to be all Moebius transformations, these are all Moebius transformations and then you have the automorphisms of the holomorphic automorphisms the complex plane are going to be just the upper triangular once and the holomorphic automorphisms of the upper half plane are going to be with real coefficients. So, but these all of course, these two are sub groups of this and they are all in any case they are Moebius transformations and the question we have to ask is what is the image of the fundamental group here the question we want to ask is what is the image of fundamental group here.

In other words we want to know which among these holomorphic automorphisms namely which among these Moebius transformations are going to be in the deck transformation group that is the question. So, we need to somehow study them in a little bit more detailed before we can cover the case when the universal cover is upper half plane.

So, let me recall what we do is that a Moebius transformation z going to a z plus b by c z plus b where ad minus bc is not 0. This Moebius transformation is identified with an element of PSL 2 c namely the normalisation that one does is that one makes the determinant equal to 1, can be normalised to satisfy determinant. So, I should write determinant of the matrix to satisfy a b minus bc is equal to 1. So, what I mean by that is you can, what I mean by that is I mean that is how we that is how we wrote the you know the groups the these groups.

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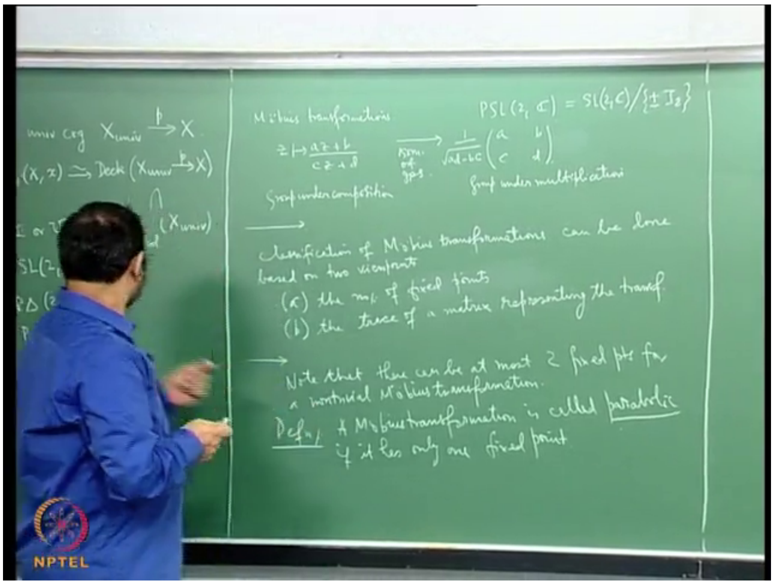

So, what we did was we took. So, for example, you take this Moebius transformation. So, this is z going to a z plus b by c z plus d and I am going to just map it to the following element of PSL 2 C, namely I put a b c d and let me call this as capital A all right I call this as capital A and what I will do is the determinant of this is now ad minus bc all right, but I would like to have the determine to be 1. So, what I do is I divide by determinant square root of determinant of A. So, what I do is I map this to the element. So, if I call this as A. So, let me write it properly. So, I will map it to 1 by root of ad minus bc times this.

So, what I mean is, so here this ad minus bc equal to 1 is a condition if we of course, if ad minus is not 1 then I will have to change this a b c d and how I change it is by dividing throughout by ad y by square root of ad minus bc because you see. Now the determinant of this is going to be you see it is going to be square of this which is 1 ad minus bc into determinant of this which is ad minus bc. So, it is determinant 1 all right. And of course, I have put a p here which means that I am not worried about plus or I am not worried about the other representative which is putting minus everywhere.

So, these two representatives are supposed to form 1 class because this is just SL 2 C the determinant 1 matrices 2 by 2 matrices modulo the subgroup given by plus or minus the identity that is what PSL 2 C is. So, let me write that on this is just SL 2 C determinant 1 modulo the subgroup given by plus or minus identity that is a 2 by 2 identity matrix. So, this is how you identify Moebius transformations with elements of PSL 2 C. And it happens that this is the Moebius transformations on this side this is the group under composition and this is also a group under multiplications standard matrix multiplication and this map is a group isomorphism. This map is an isomorphism, isomorphism of groups.

So, that is how you identify all the Moebius transformations a group of all Moebius transformations with PSL 2 C and of course, the group of all Moebius transformations is exactly the set of all possible holomorphic automorphisms of the Riemann sphere right. So, all by holomorphic maps of the Riemann’s sphere on to itself which you can also think of as by holomorphic maps of c union infinity the point at infinity on to c you in infinity all possible by holomorphic maps they are given exactly by these kind of Moebius transformations which are also called sometimes as a linear fractional transformations also as by linear transformations in some books. And the way you identify these Moebius transformations with PSL 2 C is in that way.

Now, of course, when I say the those Moebius transformations which preserves the complex plane they are going to be this subgroup and this subgroup corresponds to putting c equal to 0 you put c equal to 0 you will get the upper triangular form. And those Moebius transformations with which preserve the upper half plane the set of complex numbers with imaginary path greater than 0 the upper half plane. Then these Moebius transmissions are going to be exactly these elements with a bc and d real numbers. You will get real numbers because if a Moebius transformation preserves the upper half plane then it will also preserve the boundary of the upper half plane the boundary of the upper half plane is the real line.

So, the real line is just thought of as real axis and you know if a Moebius transformation preserves a domain it will also preserve the boundary of the domain and will also preserve the exterior of the boundary of that domain right. So, what will happen is that this Moebius if you have a Moebius transformation that preserves the upper half plane then it has to preserve it has to map the real line on to itself and that condition will tell you that all the entries have to be real numbers it will tell you that all the entries have to be real numbers. So, you can check that all right fine. So, now the question is these are the kind of objects these are the Moebius transformations that are going to come up here alright and I want to know which of them are going to be which of them are going to be coming from the fundamental group namely which of them are going to be deck transformations for a covering.

Now obviously, this has got this has to have to do with certain intrinsic properties of Moebius transformations and how we get these properties is by considering two kinds of ideas one is the fact that. So, one is based on the number of fixed points of the Moebius transformation and the other thing the other idea that helps us to classify Moebius transformations is looking at this matrix representative and looking at it is trace and the trace is you know as you know is just the sum of the diagonal elements alright.

So, what I am going to do is I am going to say the following things. So, first let me make the following definition. So, classification of Moebius transformations can be done can be done based on 2 U points the number of fixed points number of distinct fixed points or let me just say the number of fixed points or let me even say distinct and b the trace of a matrix representing the transformation.

So, the classification is done on two viewpoints. So, let me, so first of all let us look at fixed points let us look at fixed points. So, first of all notice if I have if I look at the Moebius transformation and try to study the fixed points I will get at most two fixed points that is because I will be solving a z plus b by c z plus d equal to z and that at worst is going to give me in general it is going to give me a quadratic equation in z and it will have 2 roots all right.

Of course if c is 0 I will not get a quadratic equation alright. So, note that that can be at most two fixed points for a nontrivial Moebius transformation, there can be at most two fixed points for a non-trivial Moebius transformation. In fact, if you find a Moebius transformation fixes three points or more, then it has to be the identity transformation I mean that is what this means. That is because again as I told you will at the most get a quadratic inside that you have to solve for by equating z to a z plus by c z plus d.

Now, what we do is we look at the case when the Moebius transformation has exactly 1 fixed point and we call the Moebius transformation as parabolic. So, so definition a Moebius transformation is called parabolic is if it has only one fixed point a Moebius transformation is called parabolic if it has only one fixed point right.

Now, what are the kind of Moebius transformations that you can think of which have only one fixed point. You can obviously, see that even if I take a translation z going to z plus b then that translation will have only one fixed point namely the point at infinity. So, obvious examples translations z going to z plus a let me put z going to z plus beta are certainly parabolic.

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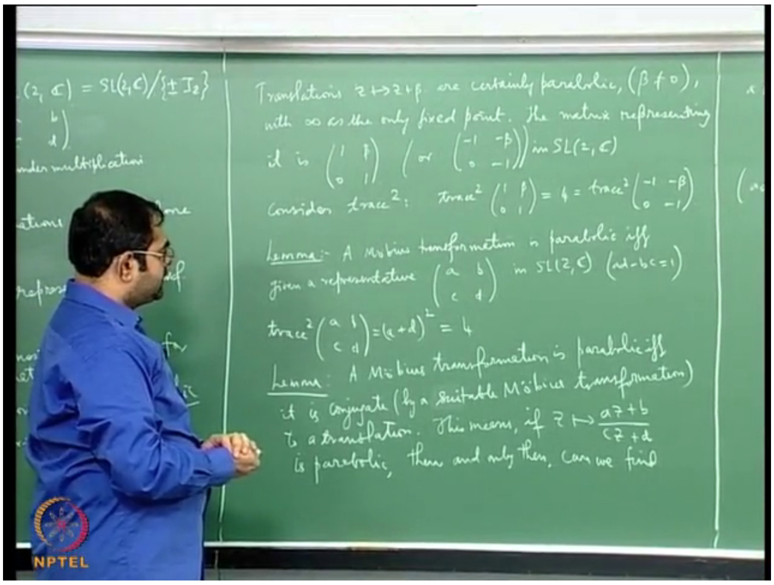

Since, of course, you know beta should not be 0 all right because I do not want the identity map I am not looking at the identity transformation, beta is not equal to 0 with a infinity as the only fixed point infinity is only fixed point.

Now, the question is as I told you the other viewpoint of looking at a Moebius transformation is by looking at the trace of a matrix that represents the transformation. So, if I take this matrix. So, what I do is I look at the matrix of this the matrix representing it, it is well it is a 1 beta 0 1 the matrix that represents that this is 1 beta 0 1 and of course, this is a determinant 1 this is determinant 1 and if I want to consider it as an element of PSL 2 C I have another choice namely I can put minus 1 minus beta 0 minus 1.

There are two representatives because if you go modulo plus or minus identity. So, this has two representatives here right. So, I let me write that r minus 1 minus beta 0 minus 1 in SL 2 C these are two representatives I can get in SL 2 C. Now well of course, you know if I take trace the trace is in this case a trace is going to be 2, trace is some of the diagonal elements, trace is 2 and in this case the trace is minus 2 you can always see that if I take an element of PSL, PSL 2 C. If I take a representative SL 2 C I will get two representatives and both the representatives the only difference will be all entries will differ in sign. Therefore, what will happen is that trace when you take the trace the trace will also change my sign. So, in order to make the make this ambiguity disappear instead of considering the trace we will consider the square of the trace. So, consider trace square. So, if I take trace squared you will see that trace square of 1 beta 0 1 is 2 and that is also is equal to 4 that is also equal to trace squared of the other representative if I take minus 1 minus beta 0 minus 1, both of them have trace squared 4.

And now it is amazing that these two properties actually characterize parabolic this property characterizes a parabolic Moebius transformation. Namely you take Moebius transformation it need not directly look like a translation, but if you know it is parabolic then it is trace will be 4 and if the trace is 4 then it will be parabolic. So, this property of the I mean the trace square. So, the trace squared being 4 characterizes a parabolic transformation trace squared being 4 characterizes a parabolic transformation right. So, let me write this lemma a Moebius transformation is parabolic if and only if given representative. So, let me write representative in the form a b c d in SL 2 C which means ad minus bc is 1 trace squared a b c d that is a plus d the whole square is actually equal to 4.

So, in other words the property of the Moebius transformation having only one fixed point is captured by the fact that you take a representative matrix in SL 2 then trace squared is 4. Of course, again you know the trace squared is important because if I take another representative the trace would have the other representative will be minus a minus b minus c minus d alright and the trace of that will be minus a minus of a plus d. So, just to eliminate that sign we take trace square and the trace squared equal to 4 is the condition for Moebius transformation to be parabolic. So, this can be kind of proved in a nice way.

And of course, let me also add one more lemma you have seen that translations or certainly parabolic. So, the converse is also true in the following sense you take a parabolic Moebius transformation then you can conjugate it by some suitable Moebius transformation so that the resulting transformation is a translation. So, let me write that lemma a Moebius transformation is parabolic if and only if it is conjugate by a Moebius transformation by a suitable Moebius transformation to a translation.

So, please try to understand the following. We have defined a Moebius transformation to be parabolic if it has only one fixed point alright and then we saw that translations are standard examples of parabolic Moebius transformations and you take a translation and you calculate it is trace square you take a translation take a representative matrix calculate trace squared you get 4. Now the trace squared being 4 for a general Moebius transformation with matrix representative forces conversely that the Moebius transmission to be parabolic namely that it has only one fixed point.

Also translations are examples of parabolic Moebius transformations and what this lemma says conversely you take a parabolic Moebius transformation you can conjugate it by a Moebius transformation and you will get a translation. So, what does this mean? This means the following. So, what this means? This means this means if z going to a z plus b by c z plus d is parabolic then and only then we can we find a Moebius transformation alpha beta gamma delta with alpha beta gamma delta and this Moebius transformation a b c d. Then multiplied by alpha beta gamma delta inverse, this is conjugating the matrix representing this Moebius transformation by another Moebius transformation. This is equal to a translation, so it should be of the form 1 0 1 beta let me put something else here. So, put t here.

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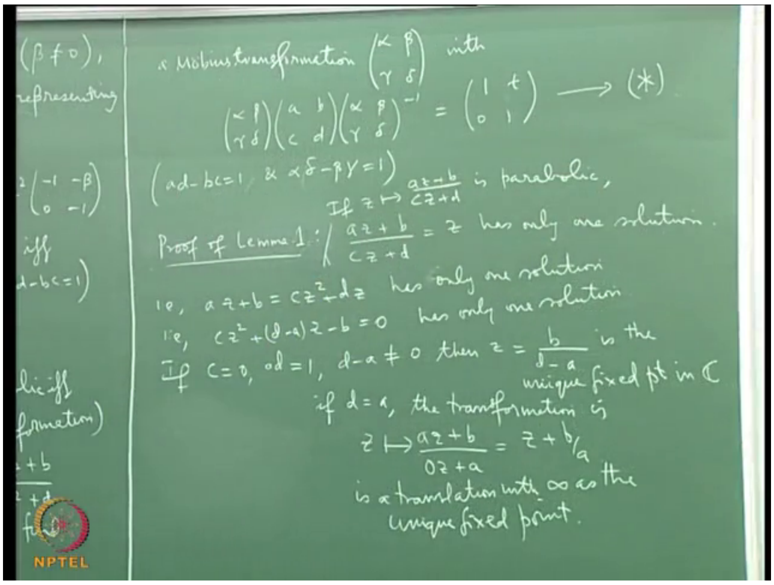

So, of course, of course, ad minus bc is 1 and a and alpha delta minus beta gamma they also 1. So, these are all Moebius transformations these are all representatives in SL 2 c determinant 1.

So, you see actually see it is very easy to see more or less that these lemmas are they are easily provable. So, for example, let me explain suppose I want suppose I have a

Moebius transformation which is so suppose you grant me the first lemma suppose you assume the first lemma is true then the second lemma can be reduced from the first.

Because you see if a Moebius transformation is parabolic then it is a by the by the by the previous lemma it is trace squared is 4 and I claim that if trace a squared is 4 you can actually solve for alpha beta gamma delta so that you get a transformation like this with for example, even with the t equal to 1. So, let me write that down. So, proof. So, let me number these lemmas let me call this as lemma 1 let me call this as lemma 2.

So, well proof of lemma 1 you see. So, you know the condition is for the Moebius transformation to be a parabolic it has to have only one fixed point. So, the condition is that a z plus b by c z plus d equal to z has only 1 solution. Then the same it has only 1 fixed point then this equation should have only 1 solution. So, this means let us write this down, maybe I should write here if z going to a z plus b by c z plus d is parabolic then this has only 1 solution that is let me cross multiply this I will get a z plus b is equal to c z squared plus d z has only 1 solution has only 1 solution and if I rewrite it I will get c z squared plus d minus a into z minus b equal to 0 has only 1 solution. So, when will that happen?

There are two cases we can consider the case when c is 0 and when c is not 0 if c is 0 then it will happen if c is 0 notice if c is 0 then ad is 1 because ad minus bc is 1 if c is 0 ad is 1. So, in particular d minus a cannot be 0 if d minus a is 0 then you will get yeah. So, let us look at this if c is 0 ad is 1. In fact, I think yeah c is 0 ad is 1 d minus a is not 0 then z equal to b by d minus a is unique fixed point.

The other possibility is d is equal to a if d is equal to a then what happens is. So, it is a c is 0 d is equal to a then the transformation becomes is z going to you see this a is going to a z plus b by c is 0. So, 0 z plus d is just a. So, I will get z plus b by a and this is a translation, this is the translation and it has only one fixed point namely the point at infinity, is a translation with infinity as the unique fixed point.

So, this disposes of the case when c equal to 0. If c is not 0 what happens. So, let me go over here and continue.

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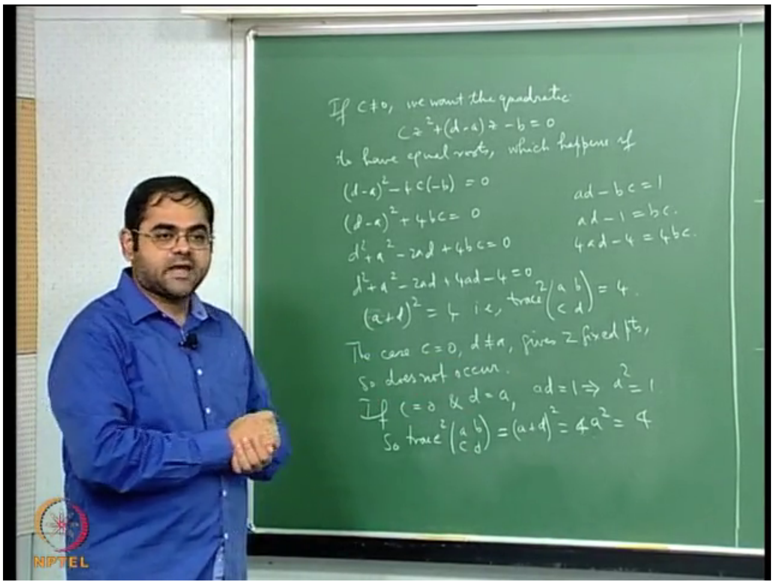

If c is not equal to 0 we want the quadratic c z squared plus it was d minus a into z minus b equal to 0 you want only one solution you want only one solution you want only one fixed point; that means, this quadratic equation should have equal roots to have equal roots which happens if the discriminant is 0. So, I will get d minus a the whole squared minus 4 into c into minus b is 0. So, the condition becomes d minus a the whole square plus 4 bc is equal to 0. If I expand it I will get d square plus a square minus 2 ad plus 4 bc is 0, but you see ad minus bc is equal to 1. So, I will replace this I will get I will get a ad minus 1 is equal to bc. So, 4 ad minus 4 is equal to 4 bc. So, if I put it back here I will get d squared plus a squared minus 2 ad plus 4 ad minus 4 0 and this will come to a plus d the whole squared is equal to 4. So, you will get trace squared a b c d is equal to 4. So, you see if c is not equal to 0 then we get trace squared equal to 4 alright.

So, I will have to go back and look at these two cases and look at whether I can conclude whether a trace squared is equal to 4. So, the first thing I want to say is you see in the case c equal to 0 this is the unique fixed point and when I say is unique fixed point it is unique fixed point in the complex plane. So, there is actually one more fixed point and that fixed point is the point at infinity and therefore, this case cannot occur because we have assumed that it is parabolic.

So, I just want to emphasize the following thing namely when I say look at the number of fixed points I mean in the extended complex plane it is important that you also look at whether infinity is a fixed point or not which is exactly happening for example, in the case of a translation in the case of a translation the unique fixed point is a point at infinity. So, you also have to look at whether infinity is a fixed point and if c is 0 then the transformation becomes z going to a z plus b by d and for that infinity is a fixed point also the point at infinity is also a fixed point.

So, you get two fixed points in c when an infinity, one of finite complex number is a fixed point the other one is a point at infinity. So, you get two fixed points and therefore, c equal to the case c equal to 0 d the case c is equal to 0 d not equal to a cannot occur. So, that case we do not have anything to prove alright. So, let me write that down. So, let me write that here in the case c is equal to 0 d not equal to a gives 2 fixed points. So, does not occur.

And what about the other remaining case, the other remaining case is when c is 0 and d is equal to a then in that case you can calculate the trace, trace squared is actually 4. If c is equal to 0 and d is equal to a then you know ad equal to 1 will imply that a squared is 1. So, trace squared a b c d is going to be a plus b the whole squared that is 4, 4 a square it is a plus a plus a the whole squared it is 2 a whole squared. So, it is 4 a square, but you know a square it is 1. So, this is equal to 4. So, you will get trace square equal to 4.

So, what we have established is that if you start with parabolic transformation which is by definition that Moebius transformation having only one fixed point in the extended complex plane then trace squared has to be 4 alright and let us prove the converse.

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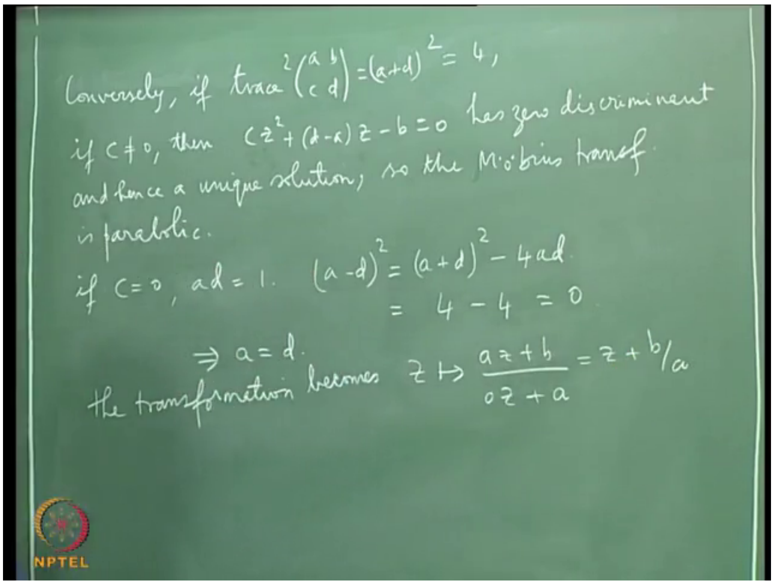

That is also quite easy to see conversely if trace square a b c d which is a plus d the whole squared is equal to 4 suppose this is the case. Again we look at the cases when c is or c is not equal to 0. If c is not equal to 0 then we are in this situation and you can actually reverse this argument and you will get that this quadratic equation has discriminant 0 and therefore, there is only 1 root and therefore, you will get that it is parabolic alright.

And notice that if, I will have to look at the situation when c is equal to 0 and the case when c is equal to 0 there are 2 cases I will show that a equal to d will occur alright and a not equality d will not occur. So, let me write that down if c is not equal to 0 then c z squared plus d minus a z minus b equal to 0 has 0 discriminant and hence a unique solution. And of course, in this case you must remember that infinity is infinity cannot be a solution for this because you have c z squared plus d minus a into z is equal to b if I push the b to other side if I substitute infinity on this side since c is not there I will get infinity on this side on the other side I will have b and b cannot be equal to infinity it is a finite complex a b c d are of course, finite complex numbers. So, in this case you will not get infinity as a fixed point there is no chance. So, this is something that you should always keep doing you must check whether by any chance infinity is becoming a fixed point.

So, if c is not equal to 0 then this has 0 discriminant and hence a unique solution. So, the Moebius transformation is parabolic is parabolic. Well if c equal to 0 we will have c is equal to 0 you will have ad equal to 1 also you will have a minus d the whole square is a plus d the whole squared minus 4 minus 4 ad and that will be, but you know a plus b the whole square is given as 4. So, I will get 4 minus 4 which is 0 and this will tell you that a is equal to d alright and, if a is equal to z d then the transformation becomes z going to a z plus b by 0 z plus a which is just z plus b by a and this has only infinity as the unique fixed point and therefore, is parabolic which has which is parabolic it is a translation.

So, of course, when c is 0 the case a not equal to d that does not occur just like we saw here. So, this proves lemma 1 alright. So, let me know try to give you proof of lemma 2 which is quite simple. So, proof of lemma 2. So, there is a very very easy case namely if you are able to find a Moebius transformation such that when you conjugate this a b c d by this Moebius transformation you get a translation suppose you are able to find 1, then you know if trace on both sides you know trace is in variant after conjugation therefore, if I compute the trace on the left side I will simply get trace of the matrix in the middle and therefore, it is going to be just a plus b the whole square.

On the other hand here if I compute the trace I get 4. So, if this holds then it is clear that the Moebius transformation has trace squared equal to 4 and by lemma 1 it has to be parabolic. So, let me write that, let me call this equation as star I am calling this equation as star.

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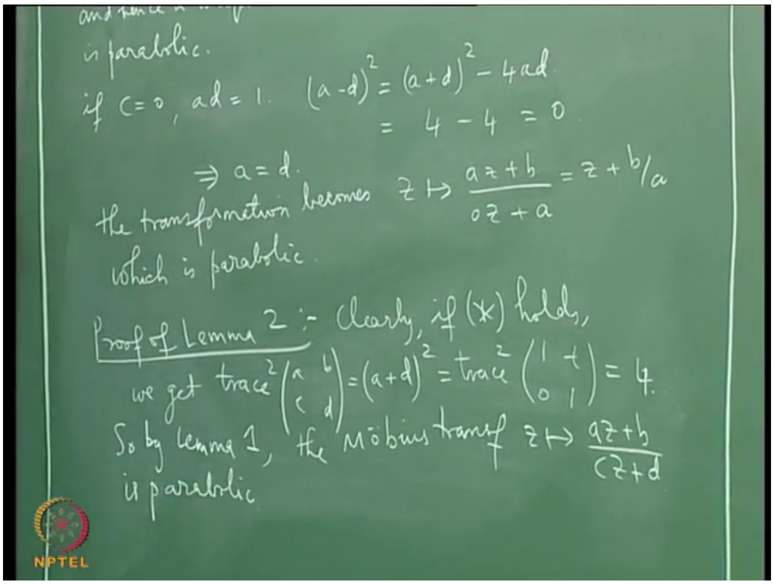

Clearly, if star holds we get trace of a plus b the whole square trace not a plus b the whole square trace square a b c d which is equal to a plus b the whole square is equal to trace squared of 1 0 1 and t which is 4, which is 4. So, by lemma 1 the given the Moebius transformation z going to a z plus b by c z plus d is a parabolic.

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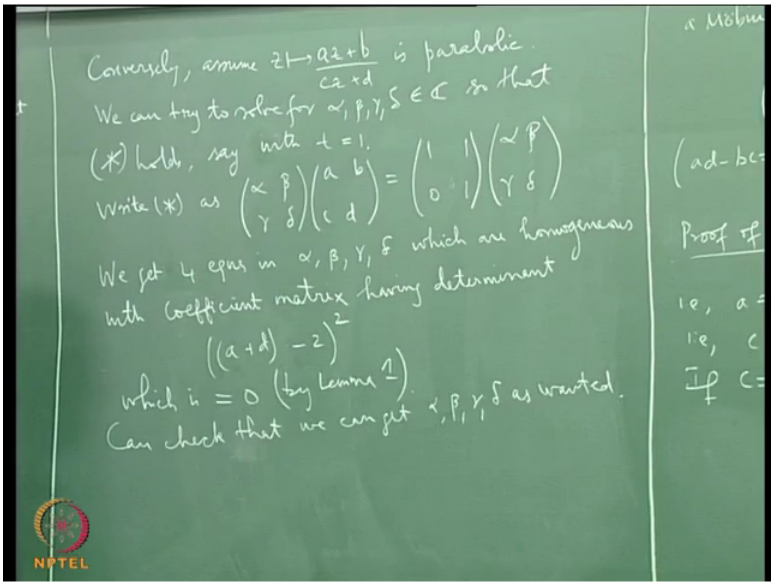

Conversely assume that it is parabolic conversely assume z going to a z plus b by c z plus d is parabolic we can try to solve for alpha beta gamma delta complex numbers. So, that star holds say t say with t equal to 1. So, you write star as you see alpha beta gamma delta into a b c d is equal to 1 1 0 1 into alpha beta gamma delta, I am just multiplying on the right by alpha beta gamma delta. We get 4 equations in alpha beta gamma delta which give which are homogenous with determinant matrix with coefficient matrix.

So, the coefficient matrix if you write it down you will get a 4 by 4 coefficient matrix having determinant and guess what the determinant is the determinant is a plus b the whole squared minus 2. So, it is no, it is a plus d minus 2 the whole squared. So, it is a plus d minus 2 the whole squared and you see which is which is not 0 not equal to 0 by lemma 1 I am sorry which is equal to 0 by lemma 1 sorry which is equal to 0. See because we have assumed this is parabolic lemma 1 says that trace squared is 4. So, trace is plus or minus 2 alright.

Now, if trace is 2 then this is 0 if trace is minus 2 you can solve for the same system by replacing a b c d with minus a minus b minus c minus d in which case you will get a minus here. So, if trace squared is 4 I can really solve for this alright and I will get a non trivial solution and therefore, I can find a Moebius transformation like this I leave you to check that you can also make sure that the non trivial solution satisfies determinant 1. So, I leave that as an exercise. So, that proves that you know lemma 2 is true.

So, you know if you want to solve a homogeneous system and you want a non trivial solution then they coefficient matrix has to have determinant 0 and the point is that the determinant has this it is connected with the trace that is the whole point.

So, I will stop here, we will continue later.