An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1 -dimensional Tori and Elliptic Curves

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Lecture - 02
Simple Examples of Riemann Surfaces

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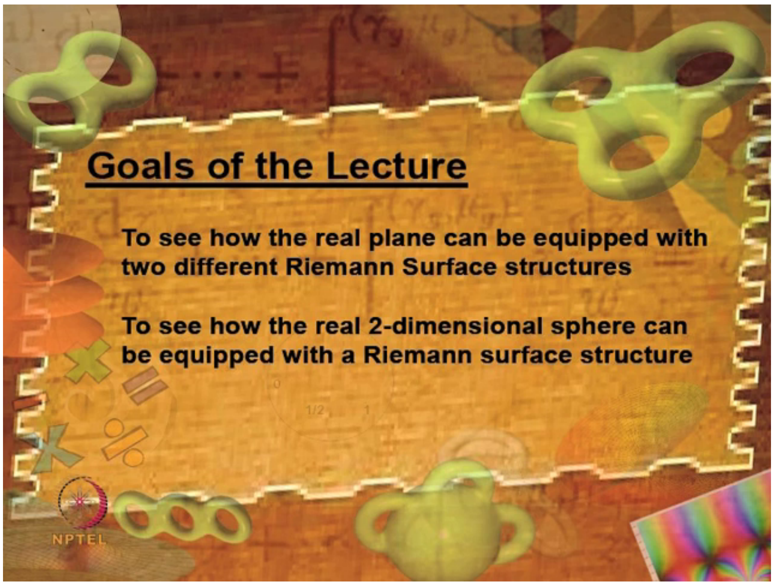

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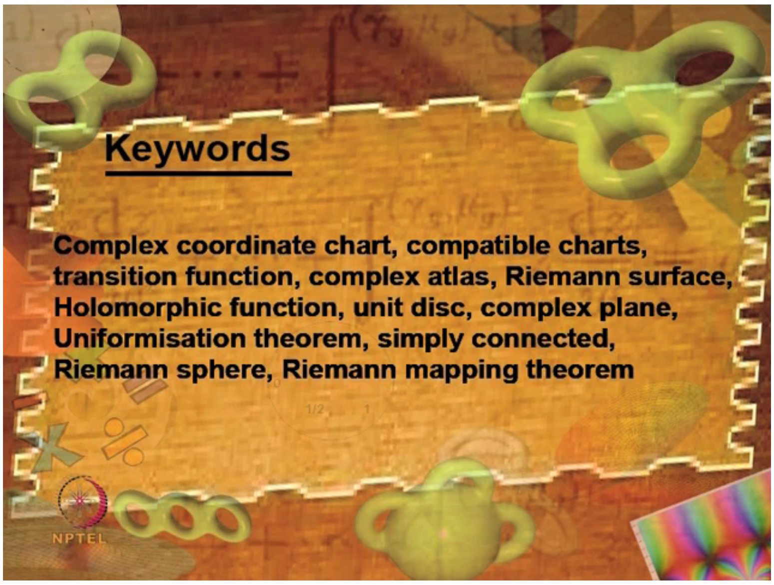

Welcome to lecture two of this course on Riemann Surfaces and Algebraic Curves. So, if you recall what we did in lecture one was to try to give the idea of what Riemann surface is and in fact, I promised in lecture 1 as 1 of the goals that I will also give some examples of Riemann surfaces which I could not do. So, I will essentially take it up in this lecture. So, let me again remind you our idea our idea is the following.

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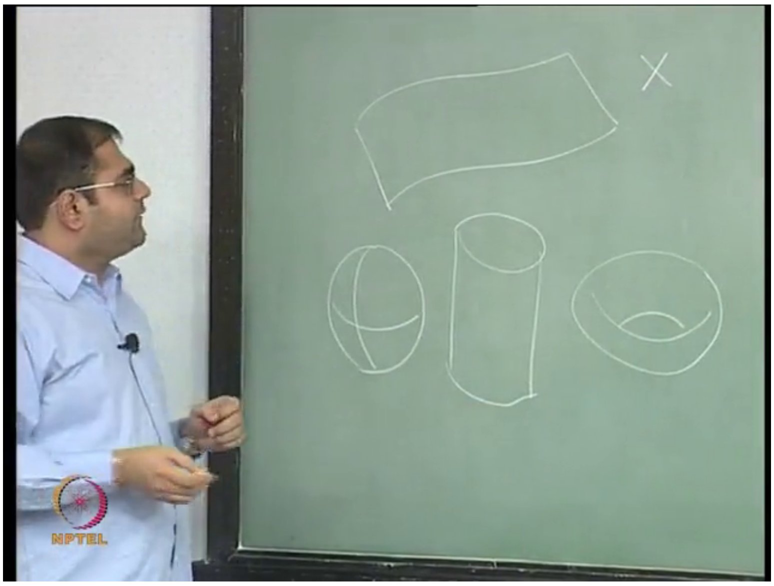

We want to start with a surface which we can visualize in 3 dimensions specific examples are of course, the sphere or the or the cylinder of course, you see what I mean by the cylinder is the infinite cylinder extending in both directions the drawn only a finite part of it here on the board and the torus.

So, these are all examples of surfaces that we can visualize in in in 3 dimensional space and what is it that we want to do we want to be able to do complex analysis on the surface. So, a Riemann surface is a structure on such a surface which allows you to do complex analysis on that surface. So, let me quickly recall.

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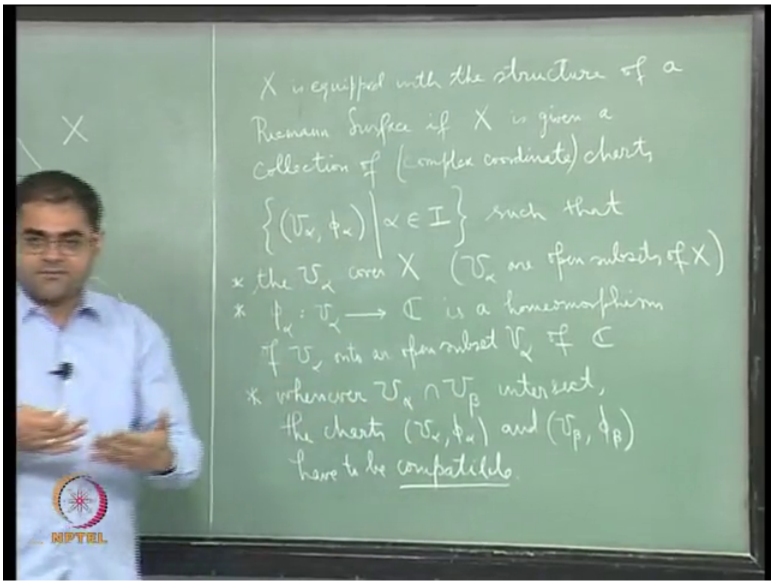

If you take if you call the surface as X, then we say that X is equipped with the structure of a Riemann surface if X is given a collection of charts of complex coordinate charts complex coordinate charts, which I will denote by pairs u alpha, phi alpha where alpha runs over an indexing set I, such that the u alpha cover X.

So, u alpha is just an open cover of X and that is the first condition the second condition is of course, I should mention here that u alpha are open sets or open subsets of X. So, you each u alpha is an open subset of X and all the u alphas for the various alphas they cover the surface, which is the same as saying that the union of the u alphas is X that is the first condition the second condition is that phi alpha is a map from u alpha into the complex plane for each alpha. Such that this map has to be a topological isomorphism that is a homeomorphism onto an open subset of the complex plane which we call as v alpha. So, phi alpha from u alpha to C is a homeomorphism of u alpha onto an open subset v alpha C and this is to be thought of as trying to give every point in u alpha this neighbourhood u alpha being parameterized by a neighbourhood on the complex plane.

So, what you are doing is by giving this homeomorphism your you are making every point in u alpha acquire a complex coordinate, because you take any point in u alpha you have a point you take the image of that point under phi alpha you get a point of v alpha and v alpha any point in v alpha has standard complex coordinates. So, this way each of these coordinate maps every chart consists of 2 data the first datum is an open set and the second datum is an identification of that open set with an open subset of the complex plane. So, the second 1 is called as a coordinate map and the whole thing together is called as a coordinate chart.

So, the essence of this is that every point in u alpha is uniquely identified with the point in the complex plane by phi alpha and then well the third condition, which is very very important is that whenever u alpha and u beta intersect, then the 2 charts u alpha comma phi alpha and u beta comma phi beta the 2 charts they have to be compatible. The charts u alpha comma phi alpha and u beta comma phi beta have to be compatible.

And what does compatibility mean I would like you to recall from the first lecture that compatibility is supposed to mean that when you try to define the holomorphicity of a function on at a point on the Riemann surface you do not want the holomorphicity to depend on the chart which you use to define which you use to give the definition of form of holomorph holomorphicity. That is there a property such as holomorphicity must be an intrinsic property it should not depend on the chart that you use.

And therefore, if a point belongs to 2 charts it belongs to 2 open sets that is in the intersection then a function at that point it is property of being holomorphic should not depend on the identification phi alpha, which is given on u alpha or on the identification phi beta which is given on u beta. It should be independent of the identifications the notion of holomorphicity should be an intrinsic property and that is achieved by this compatibility and what is this compatibility.

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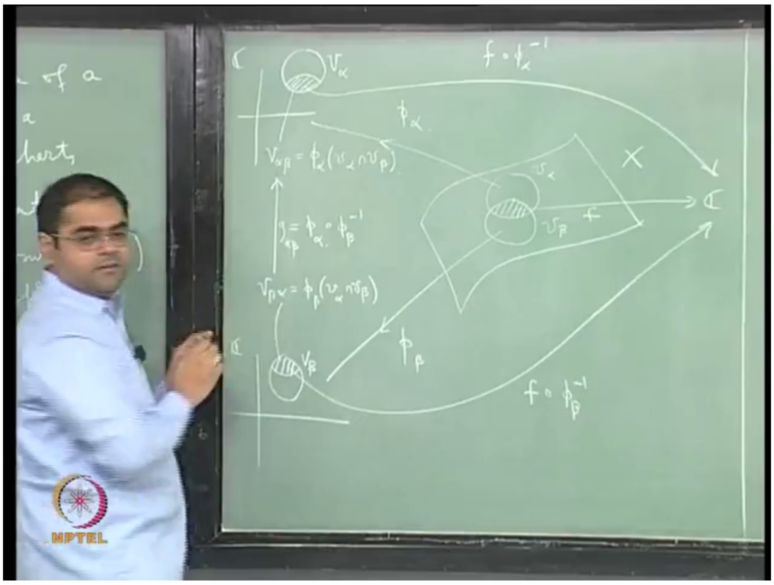

So, let me again draw a diagram to help you to recall. So, here is my u alpha and here is my u beta and well this is these are 2 open sets on the Riemann surface and we have from here this map phi alpha which is a homeomorphism on to a certain subset, which I call as v alpha in the complex plane and there is another homeomorphism v beta, which identifies u beta with again with an open subset v beta on the complex plane. Of course, when I draw these subsets need not really look like disks they could be general open sets, but I am just drawing them as disks. So, that it is easier it is easier for me to draw also well now the point is that you take this intersection which is u alpha intersection u beta that is an open subset of u alpha as well as of u beta and this intersection will be mapped by phi alpha into.

An open set here which I call as v alpha beta that is the image of that is phi alpha of u alpha intersection u beta. And similarly this open subset u alpha intersection u beta is mapped on to here an open subset of v beta by phi beta and that open subset I call it as v beta alpha, which is just phi beta of u alpha intersection u beta and what was the problem the problem was if I have a function f which is defined on this intersection and taking complex values and if I have to decide that this function is holomorphic. Then I have 2 ways of saying that the function is holomorphic. Namely, I can say f is holomorphic with respect to this identification with respect to this chart u alpha and phi alpha.

If the composite function from here to here which is given by first go by phi alpha inverse and then apply f. So, this is a composite and of course, here I mean phi alpha inverse is being applied to this subset v alpha beta. So, I am not writing it there I am just writing it just like this. So, that it is the notation does not become complicated, but remember that phi alpha is a homeomorphism of u alpha in to v alpha. So, it is also a homeomorphism of u alpha intersection u beta into v alpha beta. So, phi alpha inverse will map v alpha beta into u alpha intersection u beta.

So, well to say that f is holomorphic with respect to this chart u alpha comma phi alpha is to say that this map from an open subset by the complex plane to the complex plane this map this composition is holomorphic. And now I have another definition with respect to the other chart namely that well I can also have this composition, which is first apply phi beta inverse which will take v beta alpha into u alpha intersection u beta and then compose it follow it up with f. And I can now say that f is holomorphic with respect to this identification if f circle phi beta inverse is holomorphic. And what I really do not want to happen is that it should not be that the f is holomorphic with respect to 1 chart and not holomorphic with respect to the other chart; that is it should not happen that this is holomorphic, but this is not holomorphic I do not want such a conflict and that is, because the idea of holomorphicity of a function should be intrinsic to the function it should not depend on any external factors.

So, it is something like this in linear algebra for example, if you have a finite dimensional vector space you see no matter what your basis is you do not expect the dimension to change the cardinality of a any 2 bases are the same . So, it should not be that that is because the dimension of a space is something that is intrinsic it should depend only on the space and not the way you get at it and that is why we have the theorem that if you take any 2 bases they have the same cardinality. So, you see this intrinsic thing in mathematics they should be defined in such a way that is ambiguous is not ambiguous.

So, holophorphicity is an intrinsic thing and you do not want it to be and ambiguous and ambiguity will come in the moment you have 2 charts any 2 intersecting charts and a function defined on the intersection of those 2 charts. So, how did we X get pass this condition this compatibility is given by the following we require? So, I let me say compatibility requires that.

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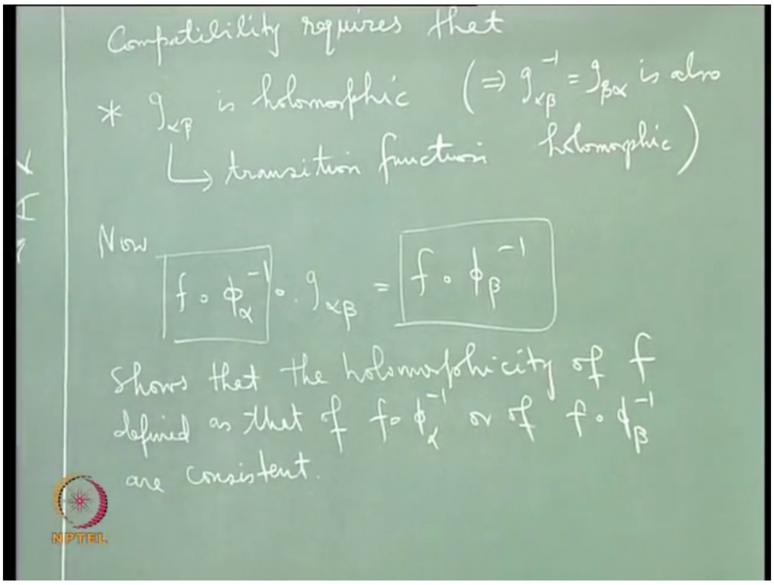

So, I look at this map namely I go from v alpha v beta alpha and then I go all the way to v alpha beta. So, I take this map there is a map like this how do I get this map this map is well first apply phi beta inverse that will take v beta alpha into u alpha into u beta and then follow it by phi alpha.

And well you can call this as g alpha beta and of course, this g alpha beta is a homeomorphism because phi beta inverse is a homeomorphism and phi alpha is a homeomorphism. So, it is a homeomorphism and, but you must understand that all this is only defined on v beta alpha and it lands in v alpha beta. So, if I had to be very strict here I have to write phi beta inverse restricted to v beta alpha and then instead of phi alpha I should write phi alpha restricted to u alpha intersection u beta I am not writing those restrictions, because then it will look very complicated. So, what is the compatibility requirement is it is that this g alpha beta is holomorphic this is the condition.

So, the compatibility of any 2 charts that whenever there are 2 charts whose domains intersect then the compatibility condition is that this this function that you can write down for these 2 charts which is called the transition function should be holomorphic. So, this g alpha beta is called the transition function and you put this condition that g alpha beta is holomorphic then it becomes very nice it is it becomes it is also enough to guarantee that g alpha beta is actually an isomorphism holomorphic isomorphism because it is already a homeomorphism. So, it is injective and I told you that an injective holomorphic map is an isomorphism onto it is image and the inverse map is also holomorphic. So, this g alpha beta is holomorphic will automatically imply that g alpha beta inverse which you can check is g beta alpha is also holomorphic.

So, you get this also if you reverse roles of alpha and beta you can similarly define g beta alpha and you will find that g beta alpha is nothing, but g alpha beta inverse and well and why is it that this compatibility is going to help us it is going to help us because you see if I take. Now if I look at f circle phi alpha inverse and if I compose it with g alpha beta I will end up with f circle phi beta inverse you see. So, I have this expression and what does this expression say it says very clearly that since this g alpha beta is an isomorphism it says that this is holomorphic if and only that is holomorphic. And therefore, the holomorphicity of this is equivalent to the holomorphicity of that and that is exactly what I want the holomorphicity to if it is holomorphic with respect to 1 chart. Then I want that it should also be holomorphic with respect to any other chart which intersects this chart.

So, that condition is the compatibility condition and that compatibility condition ensures that the holomorphic nature of a function is intrinsic it does not depend on the chart that is the whole point and that is that is where these. So, called transition functions come in so shows. So, this equation shows that the holomorphiity of f defined as that of f circle phi alpha inverse or of f circle phi beta inverse are consistent. So, you define f to be a holomorphic that if f circle phi alpha inverse is holomorphic or you define f to be holomorphic if f circle will be phi beta inverse is holomorphic; these 3 these 2 definitions are going to be consistent. That is because the difference is captured by a transition function which I have made it which I have required it to be a holomorphic isomorphism fine.

So, let me again tell you a very quickly that when you have this this collection of charts which consists of consists of Parawise compatible charts. So, you must notice that compatibility has to be checked only when 2 charts intersect. So, for any 2 charts that intersect. So, whenever any 2 charts intersect if they are compatible and if I have collection of charts that cover then I say that X is now a Riemann surface, I say that the surface X has been given the structure of Riemann surface.

Now, this collection of charts this collection of compatible charts has a special name for it is called an atlas.

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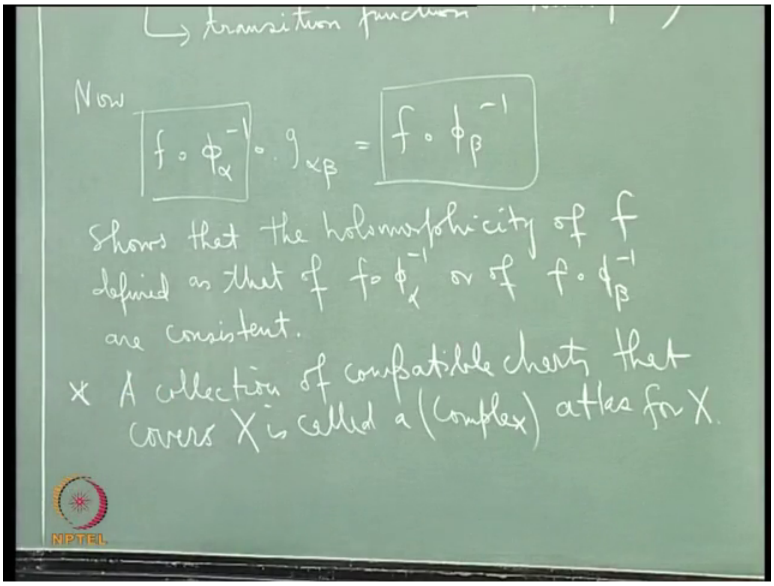

It is called a complex atlas. So, a collection of compatible charts that that covers X is called a complex atlas for X. And so now, you can see it in a jiffy that the surface X is endowed with a structure of Riemann surface if you can find for it complex atlas. So, our definition of Riemann surface structure on a surface on a surface X is just reduced to finding a an complex atlas for that surface.

So, now having said this of course, this definition needs to be improved a little bit more formally. But essentially this is the content of what Riemann surface is. So, I will get into details in the coming lectures. So, there are certain conditions for example, I must always assume X to be connected and if I am assuming X to be a surface which is an abstract surface then, I will have to define what an abstract surface is. So, what you must understand is that at the moment I am just thinking of surfaces which I can really visualize in 3 dimensions and I am thinking of trying to make them into Riemann surfaces, but then if I want to take an abstract surface and make it into a Riemann surface then I will have to first define what an abstract surface is.

So, I will have to define what a 2. I have to make sure that this abstract surface is something that is 2 dimensional because the surface is always 2 dimensional and then this will lead me into some technicalities. So, I am such a definition is possible, but I do not want to get into that now we will get into that in the later lectures. So, for the moment let us take this working definition and try to look at some examples. So, there are some examples and connected with these examples there are some theorems which are really striking and which are deep theorems, but already they are quite striking.

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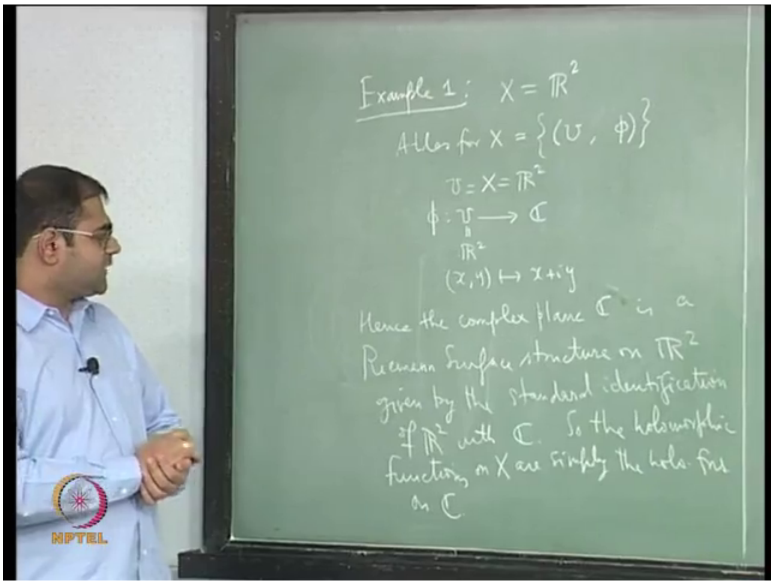

So, let us look at some examples. So, here is here is example 1 here is example 1 I take X to be the real plane the X Y plane and well I take the atlas for X to be consisting of just 1 chart. So, you know every chart contains an open set and a map all right and the open set I take it to be all of X that is I take it to be all of R 2, I take the whole plane and the map phi mind you has to be a map from u into C and, but u is of course, R 2. So, this map from R 2 to C what is a map it is a natural map it an identification map and what is that map it is just taking x comma y to x plus i y. So, this is the natural identification of the plane of the real plane with the Argand plane.

This is a natural identification and of course, this map is homeomorphism this is this of course, a continuous and the inverse is also continuous is the identity map. So, essentially this is the identity map from R 2 to R 2 the target R 2 being thought of as an complex the complex plane. So, this is homomorphism of course, and therefore, this pair u comma phi is a chart and you see it is it covers X because the domain of the chart is already all of X. So, 1 chart is enough 1 chart is enough and well to say that this is an atlas I should also check the so called compatibility condition.

But then there is no compatibility condition need that needs to be check here because the compatibility condition needs to be checked only when you have 2 charts which intersect and there is only 1 chart. So, there is nothing to check. So, it is vacuously true. So, this becomes a complex atlas and what happens is that well with this complex atlas R 2 becomes a Riemann surface. So, with this complex atlas are 2 becomes a Riemann surface and what is that Riemann surface it is just the complex plane. So, that. So, we say that the complex plane is Riemann surface structure and R 2 given by the natural identification this is the natural identification. So, hence the complex plane C 2 is a Riemann surface structure on R 2 given by the standard identification of R 2 with so and well what is the meaning of a holomorphic function?

So, a function f on the Riemann surface is going to be a function of 2 variables real variables x and y, if you call it as f of x comma y when is it holomorphic it is holomorphic if you write x plus I y as z and express that function as a function of z it has to be holomorphic. So, it is a function on R 2 that is holomorphic is actually a holomorphic function in the usual sense. So, the holomorphic functions on X are simply the holomorphic functions on complex plane.

So, if a want to expand on that, let me rub this side just for the sake of clarity. So, if I have from so my X is R 2.

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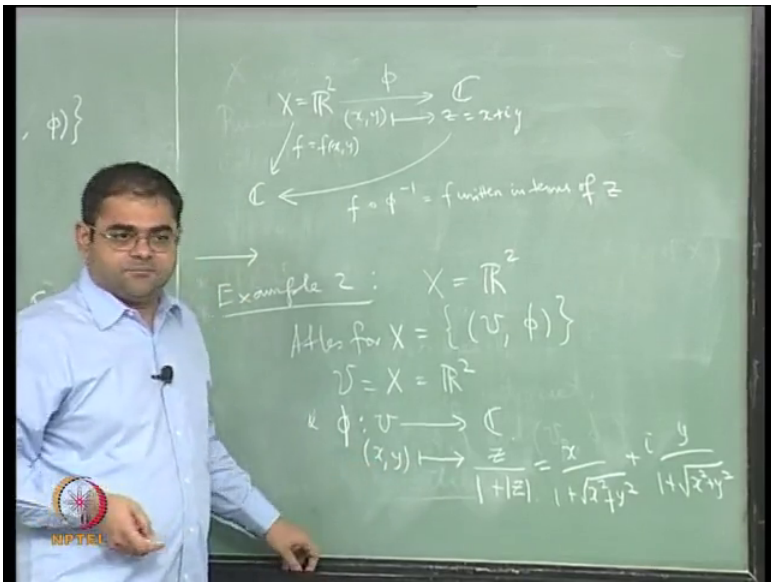

And suppose I have an f a function f into c which is given by f is equal to f of x comma y for x comma y of point in R 2 when do I say it is holomorphic by my definition of holomorphicity well I take this this chart, that identifies x comma y with z with z is a x plus i y and then I have to take this composition. If I take this composition what I will get is it just trying to write f as a function of z f here is written as a function of x and y and if I take this composition this composition is to be very strict I should write phi inverse followed by f.

So, let me rub this out and let me write it very clearly. So, I have phi inverse followed by f if I write this I get that is nothing, but f written in terms of z that z is x plus i y I am saying that this f is holomorphic is the same as trying to say that write if as a function of z and it should be holomorphic. So, it is a usual definition it is a usual definition you do not get anything new. So, this is a simplest example right you had a question.

Student: (Refer Time: 27:56) c is a (Refer Time: 27:58).

This should be c sorry thanks that should be C C 2 is not the complex plane thank you. So, well now we can ask the following question. So, you look at this line very carefully it says well the complex plane C is a Riemann surface structure on R 2. So, look at this a Riemann surface structure. So, the question comes can I give to R 2 some other Riemann surface structure, which is different from the usual complex plane structure and that is what the second example is going to tell you the answer to that is yes. So, let us look at the second example. So, here is my example. So, again my X is R 2 it is a real plane and well the atlas for X is again consisting of only 1 chart u comma phi only 1 complex coordinate chart and where u is of course, X and which is R 2 and phi.

Now, is going to be slightly different phi is going to be from u to c. So, if I take any x comma y what is it going to go to well here I send x comma y to x plus i y which is z. So, now, what I am going to do is I am going to send it to z by 1 plus mod z. So, which means what it means is I am going to send it to x by 1 plus root of x square plus y square plus i times y by 1 plus root of x square root y square this is what I am going to send it to you. So, here is my map my map from u to c is the map that sense x comma y to this complex number right. So, the complex number is z by 1 plus mod z, but z is the usual exposit. So, I am not sending x comma y to the to z, but do something else now you can check that this is a homomorphous it is easy to see. So, let me rub of this diagram you have a question.

Student: Modulus of the function value will be always less than or equal to 1 modulus of (Refer Time: 30:48).

Exactly.

Student: So, how it will be a homeomorphism.

No mind you a coordinate map is; supposed to be only a homeomorphism onto the image, which is an open set.

Student: (Refer Time: 30:59).

So, it should be a when you take a coordinate map it should be from an open subset of the surface to an open subset of the complex plane which you need not be the whole of the complex plane . So, now, what I have done is you know what I am trying to do I am trying to map the whole complex plane onto the unit disk open unit disk.

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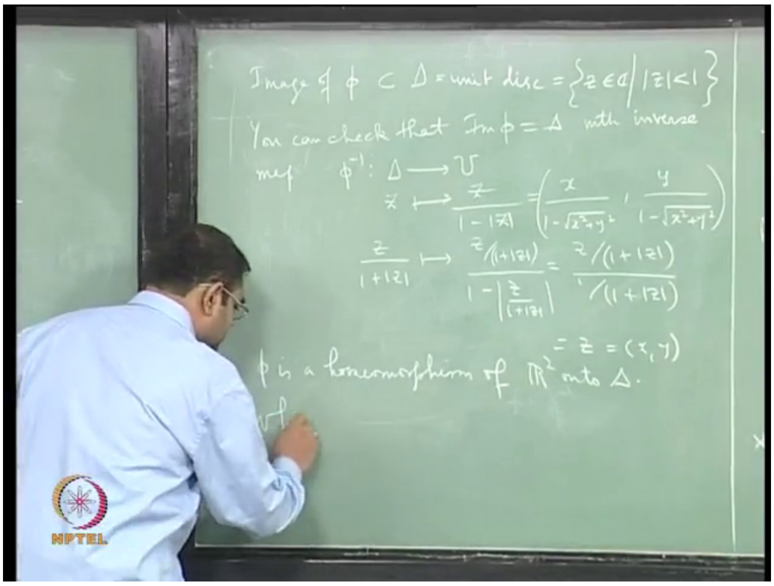

You see that is what is happening. So, you see you can see the image of phi is contained in the unit disk, which is given by the set of all complex numbers z whose modulus is strictly less than 1 that is because if I take the modulus of this complex number z by 1 plus mod z that is always going to be less than 1.

So, the image of this course in the unit disk and in fact, the and in fact, you can check you can check that the image of phi is actually the unit disk with inverse with inverse map phi inverse this is from the disk map to u mind you use R 2 and what is a map it is just you send z to I think u z by 1 minus so z. So, let me just check it for a moment z by 1 minus mod z. So, this is the map and when I write it like this I mean that you will have to be you have to change this back into x y so; that means, you should write this as x by 1 minus root of x square plus y square comma y by 1 minus root of x square plus y square.

So this is the inverse map and you can check that this is indeed this is indeed the inverse map because you see I start with x comma y it goes to z by 1 plus mod z where I take whereas, as usual I take z equal to x plus iy. Now if I take z plus z by 1 plus mod z it is here instead of z if I replace z by z by 1 plus mod z then this expression will simplify to z you can see that. So, z by 1 plus mod z we will therefore, go to z by 1 plus mod z by 1 minus modulus of z by 1 plus mod z if I simplify this I will get. So, this will be z by 1 plus mod z. So, let me put a bracket around it. So, that there is no confusion and here if I simplify this I will get 1 by 1 plus mod z so that will be equal to z and that is your x comma y.

So, it is indeed the inverse map. So, what I have proved is that this map phi is a homeomorphism of the whole plane on to the unit disk phi is a homeomorphism of R 2 onto the onto delta. So, well again I am in good shape see I have 1 chart and therefore, it is an atlas because I do not have to worry about the compatibility. So, I have got a Riemann surface, now is this Riemann surface is the same as the complex plane it is not, because why is it not the same as a complex plane because let us try to understand what is a holomorphic function on this Riemann surface.

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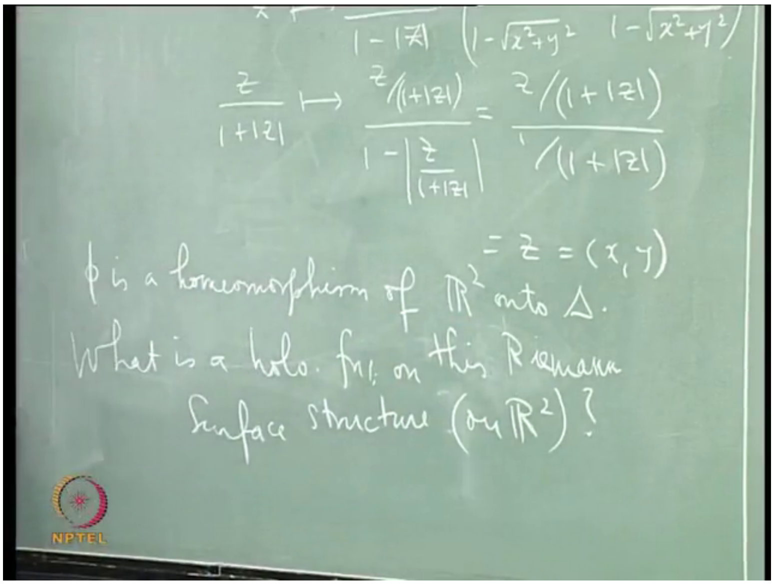

What is a holomorphic function on this Riemann surface structure? What is the holomorphic function on this Riemann surface structure on this; on this Riemann surface structure on R 2 what is a holomorphic function. So, let us go back to the definition and you will be surprised and I am hoping that it will remind you of an important theorem in in in complex function theory.

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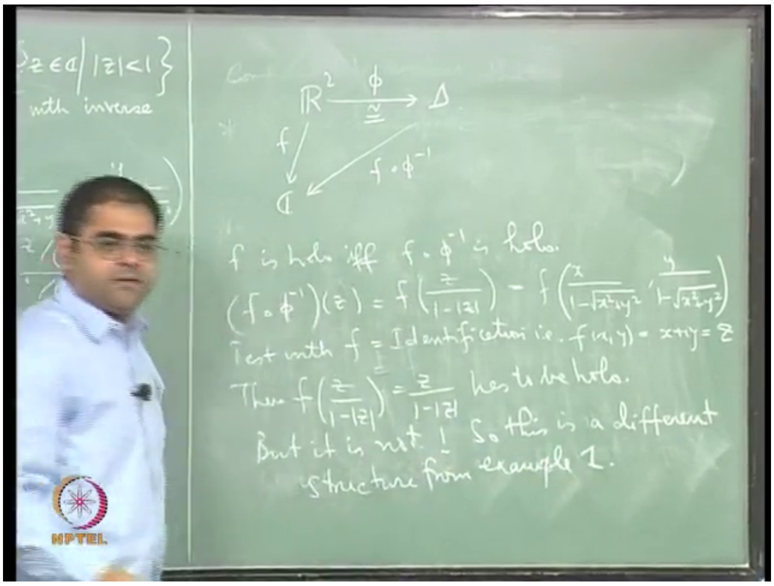

So, you see you have. So, you know. So, here is my Riemann surface it is R 2 my chart is from this into the unit disk this is my homeomorphism. So, I put this this symbol to say that it is a homeomorphism or if that confuses you maybe, but that is now.

Suppose I have a function f a complex valued function when is when is f is holomorphic if and only if this composition which is first apply phi inverse followed by f is holomorphic, but what is phi inverse followed by f see f circle phi inverse of z is f of phi inverse of z, but I have a form of a phi inverse of z it is f of z by 1 minus mod z. So, it is f of z by 1 minus mod z and well what is this going to be this is going to be and well none if I want to be. So, what it means is that my f is just now f is just a function of x and y. So, I will have to write this f of well x by 1 minus root of x square plus y square comma y by 1 minus root of x square plus y square.

So, I should write f like this. So, you see what has happened the function now is not the function x comma y going to f of x comma y, because of this chart it has become the function the x comma y going to f of x by 1 minus root of x square plus y square y by 1 minus root of x square comma x y it is no longer the same function x comma go y going to f of x comma y. And if this is a holomorphic I want this to be holomorphic I want this to be holomaorphic. So, you know this is holomorphic in if and only if this function is holomorphic, but then let us test with f is equal to the identity map identification, which is just that is that is f of x comma y is equal to x plus I y that is z.

This is just the natural identification map. So, then you see you will get then you will get f then the condition will be that f of z by 1 minus mod z, it will be simply z by 1 minus mod z right and then this has to be holomorphic, but it is not f of z going to z by 1 minus mod z is not holomorphic that is, because you can see that this mod z is square root of z z bar and you know that the moment a z bar term comes the function cannot be holomorphic the partial derivative of f with respect to set bar has to be 0. So, z going to z by 1 minus mod z is not holomorphic. So, what you have done is on R 2 you have put a Riemann surface structure such that the natural identity map becomes non holomorphic.

So, you see it is a completely differently structure. So, this structure of Riemann surface of R 2 is not the standard structure it is a new structure and guess what it is nothing, but it is a Riemann surface structure on the unit disk. After all you are identifying R 2 with the unit disk and it is the complex structure on the unit disk that you are transpose transporting to R 2. So, what you have done is you have made R 2 into a Riemann surface isomorphic to the unit disk all right, but then you know the unit disk is not equal into the complex plane that is your famous Riemann mapping theorem it says that any, if you look at the open simply connected subsets of the complex plane there are only 2 isomorphism classes 1 is the whole plane and anything else is isomorphic to the unit disk and these 2 are different.

So, what I have done is by this map I have forced the structure of Riemann surface of the unit disk to be stretched to the whole plane and so it is not this standard structure on the whole complex plane. So, that is why I am getting another structure on the compet on R 2 which is not the standard structure. So, but it is not. So, let me just complete it. So, this is a different structure from example 1. So, the remark is that you can still look at the situation when I mean the situation that is described by the Riemann mapping theorem. So, let us now go into the next example which is I will trying to make which is trying to make the sphere into Riemann surface.

So, before I before I do that let me add a remark. So, this is the probably the right point to add the following remark. So, the remark is a very deep theorem. So, it is called the theorem it is part of what is called the as the uniformization theorem.

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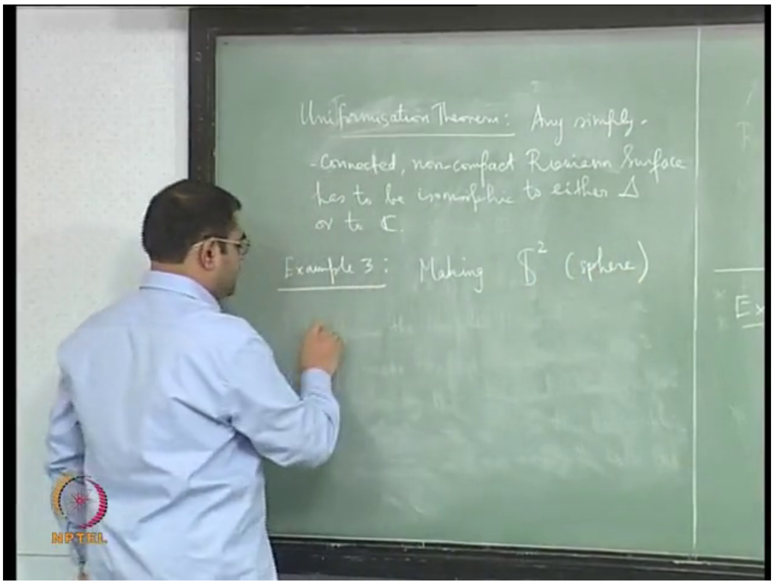

So, let me state that. So, uniformization theorem any simply connected non compact Riemann surface has to be isomorphic to either the unit disk or to the whole complex plane. So, what this will tell you is that if I try to attempt to put various Riemann surface structures on the plane I will only succeed in getting 2 which are essentially different 1 will be the whole complex plane itself the standard structure.

The other 1 is the structure that I have written down here you cannot get anything more well the proof of this theorem is a little deep and essentially it can be in a way reduced to the Riemann mapping theorem all right, but I am in this course I am trying to I am going to point out some important theorems at the right time not to be worried about giving the proof of the theorem immediately, but you just it to tell you what is true for the. So, that you get a feel of what is true. So, what this if you believe this uniformizing zation theorem it is not the full uniformization theorem it is still a part of the uinformizatiion theorem.

So, here what I have said is I have said that the Riemann surface should be connected on non-compact. Now of course, you know what connected means topologically that it cannot be written as a disjoint union of open sets simply connected of course, means that any closed loop can be continuously shrunk to a point which means that there are no holes on the surface and non-compact is also something that you know the condition is that it should not be closed and bounded if you if you visualize it as a subset of R 3.

So, well this is the uniformization theorem and if you believe this it is very clear that you can deduce that you see the the only 2 possible Riemann surface structures that we can force on the complex plane are 1 or the complex plane itself which is given by example 1 by the standard identification and the second 1 is the Riemann surface structure on the unit disk these are the only 2 possibilities you cannot get any more.

In other words you try to force any Riemann structure on R 2 it will either by some of it to see as a Riemann surface are to be isomorphic 2 the unit disk as Riemann surface that is all you will get. So, it is a deep theorem now let me go to example 3; which is trying to make this sphere into a Riemann surface. So, make making a sphere sphere into Riemann surface well.

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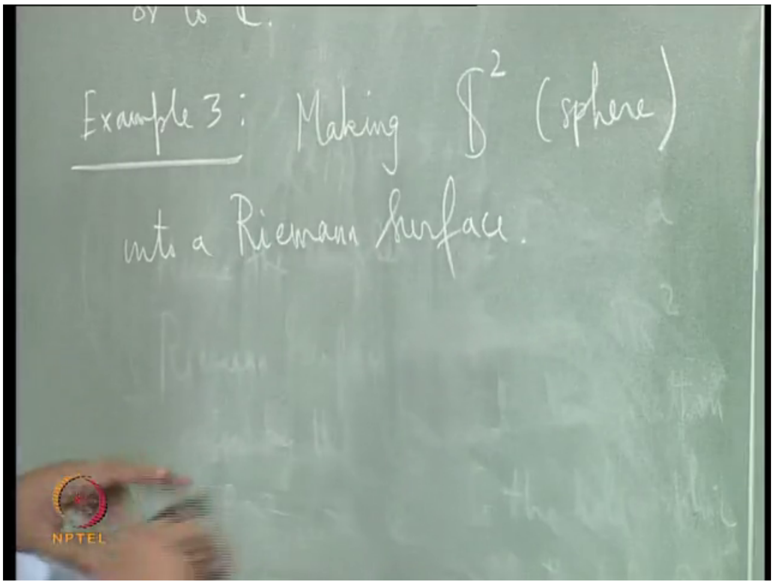

So, here I am. So, let me let me draw a diagram. So, that it is it is easier for me to explain how the charts look like.

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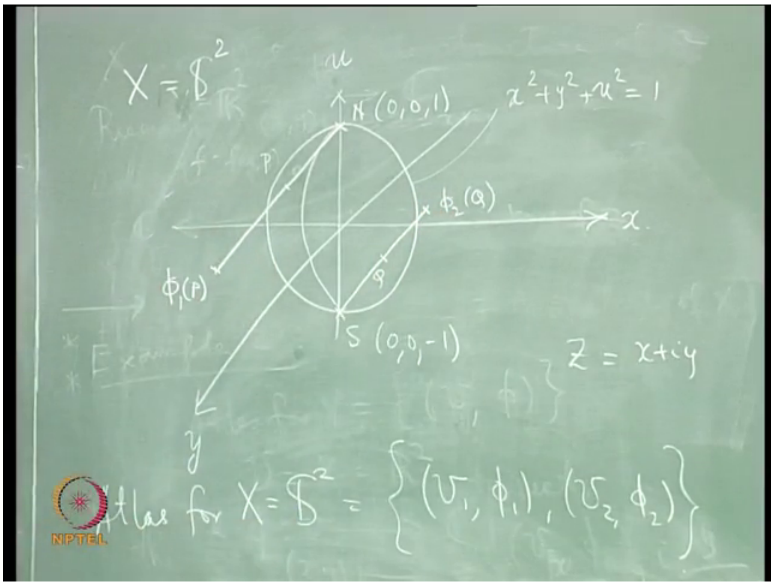

So, I take 3 dimensional space and well I have the unit sphere here in 3 dimensional space. So, this is the unit sphere I call this the this is my x axis this is my y axis well and I want call the third 1 as the z axis I call it as a u axis because I want z to be x plus i y that is I want the x y plane to still represent the complex plane and that is why I am not calling the third axis is the z axis. So, I am calling the third coordinate as u and well this is this is the sphere s 2 and of course, you know this is the North Pole n this is the South Pole and you know the coordinates the North Pole is 0 comma 0 comma 1 South Pole has coordinates 0 comma 0 comma minus 1 and well to make this to make the sphere into a Riemann surface. I have to give you an atlas an atlas is therefore, going to be collection of charts which are parawise compared to be whenever they intersect and what I am going to do now is give you an atlas consisting of exactly 2 charts and what are the 2 charts 2 charts are as follows.

So, atlas for X equal to S 2. So, this is my X now is it consists of let me write this u 1 comma phi 1 and u 2 comma phi 2 and let me tell you what you want you to phi 1 phi 2 are well.

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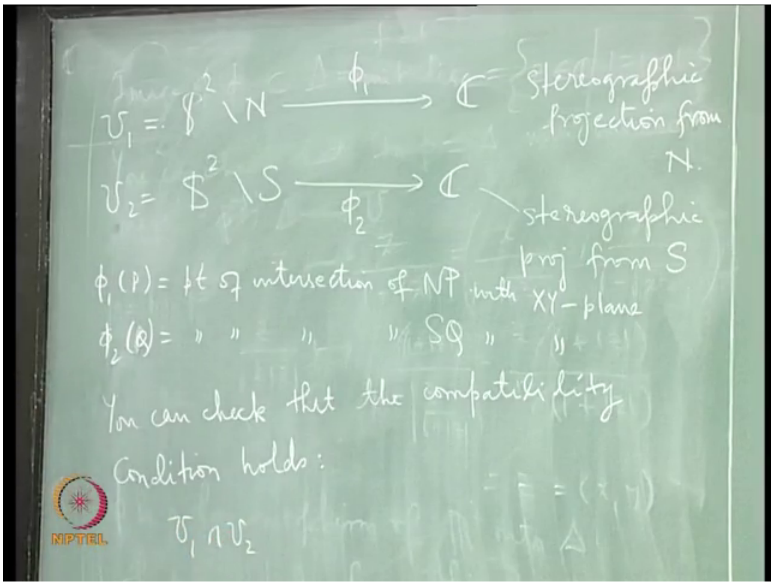

So, u 1 is this sphere minus the North Pole and u 2 is a sphere minus South Pole and the map phi 1 from u 1 to the complex plane is nothing, but the stereographic projection from the North Pole. So, this is the stereographic projection from N. And similarly the map from phi 2 to C is going to be the stereographic projection from the south. So, well let me quickly recall this u I am sure you have seen this in a first course on complex analysis how the Riemann sphere is constructed. So, it is well you take any point on the sphere p and what are the stereographic projection from the North Pole well you just join this point to the point p you get a straight line and that line is going to go and hit the complex plane which is the x y plane thought of as a complex plane.

At a certain point let me call this as at a certain point which I call as phi 1 of p that is a stereographic position for the north and similarly what is the stereographic projection from a South Pole well you again you take a point Q on the sphere the reimanns sphere mind you this is given by the algebraic equation x square plus y square plus u square equal to 1 of course, just to recall. So, I am taking a point Q with coordinates which satisfy this equation and well what is the stereographic position I joined the South Pole to this point and it is going to come up I am going to hit the plane somewhere and I call that as phi 2 of Q.

So, you see in other words phi 1 of p is equal to interest point of point of intersection of NP or PN with X Y plane and similarly phi 2 of p phi 2 of Q is point is a point of intersection of S Q with the X Y plane. So, these are the standard stereographic projections and well you can easily check you have already seen this I suppose in a course in a course in a complex analysis in the first course in complex analysis that both stereographic projections are Isomorphisms they are homeomorphisms of the Riemann sphere minus the pole onto the whole complex plane. So, these are of course, homeomorphisms.

So, both u 1 comma phi 1 and u 2 comma phi 2 are certainly charts and u 1 and u 2 of course, cover the whole sphere. So, you have charts that cover the whole sphere the only thing that you have to worry about it is to make this really an atlas you have to check that the compatibility condition course. Well, you can check that the compatibility condition holds and what is the compatibility condition you take u 1 intersection u 2 and so the compatibility condition will come from so, u 1 intersection u 2 will be the Riemann sphere minus both the North Pole and the South Pole.

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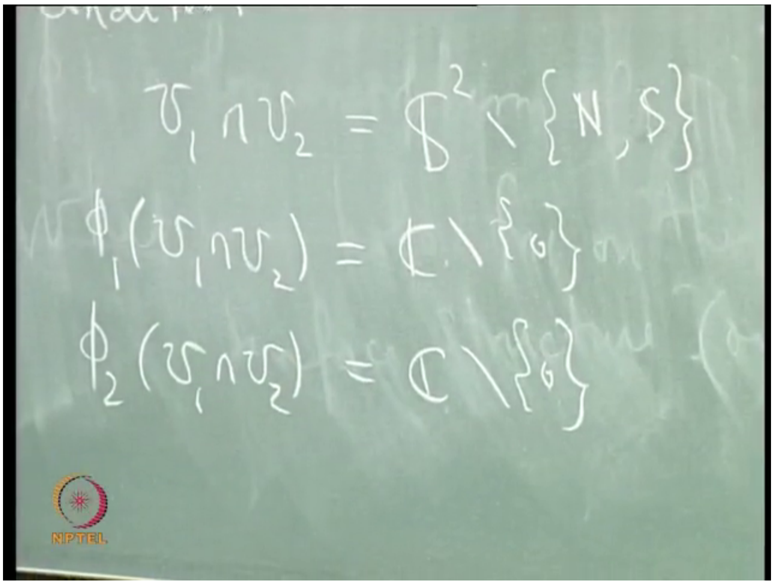

And you can see that phi 1 of u 1 intersection u 2 is just the complex plane minus the origin and because phi 1 is going to map the whole sphere except the North Pole on the complex plane. And if you remove the South Pole then from the image you are removing the origin and you can see also that phi 2. Similarly of u 1 intersection u 2 is again the complex plane minus the origin.

So, in both cases the image of this intersection is the. So, called punctured plane the plane minus the origin and therefore, you get a transition function from the puncture plane to itself, and I will have to check that this transition function is holomorphic. And you can check that the transition function is has a very simple form in this case namely.

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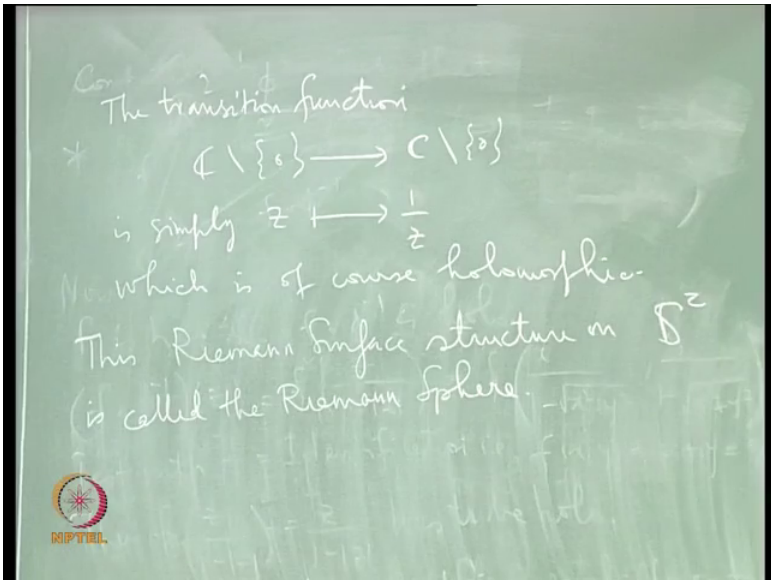

The transition function which we will go from C minus 0 to C minus 0 the transition function is simply z going to 1 by z you can check that you can I want you to write it down and check that this is the transition function and this is of course, holomorphic is of course, which is simply which is of course, holomorphic well as a result of this we have been able to give a Riemann surface structure on the sphere well coming to think of it probably you will have to compose either phi 1 or phi 2 with the complex conjugation to get it right.

So, you have to figure that out writing it down to yourselves and this Riemann surface structure on the sphere is called the Riemann sphere this is what. So, this this Riemann surface structure on s 2 is called the Riemann sphere. Well, let me end by giving you another interesting result see in the first and first and second examples we saw that we were able to give 2 different Riemann surface structures on the plane and then the uniformization theorem said that these are the only ones possible.

Now you can ask the same question take the real sphere I have already proved that there is 1 Riemann surface structure can you give more. So, there is a theorem it is again part of the uniformization theorem which says that this is not possible you can on this on the on the real sphere S 2 any structure of Riemann surface that you impose on it will be isomorphic to this 1.

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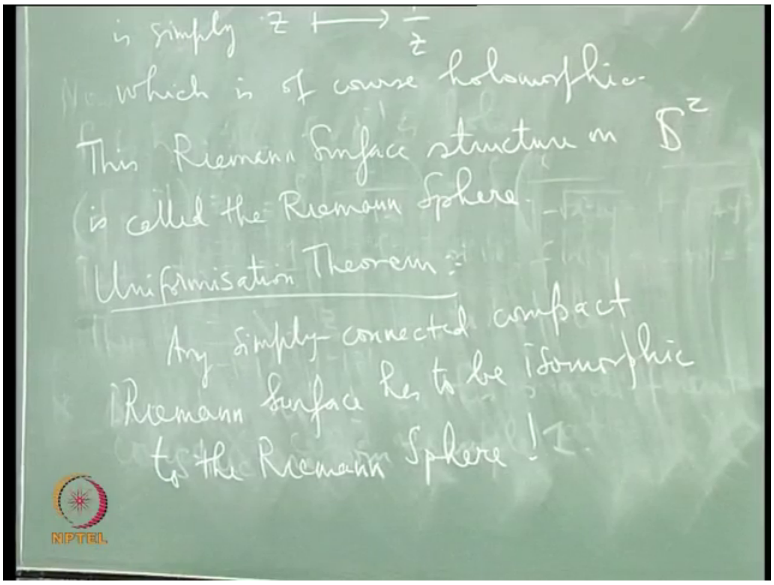

So, you have no freedom. So, this is also. So, let me state this uniformization theorem any simply connected compact Riemann surface has to be isomorphic to the Riemann sphere. So, it is a beautiful theorem it tells you that you no matter what no matter how many charts you use and no matter how many manor atlases you try to manufacture on the real sphere if you put a Riemann surface structure it has to be isomorphic to this 1 you do not have any more freedom.

So, this is a again a deep theorem it is part of the uniformization theorem this and that statement put together are called as a first uniformization theorem this is for non-compact simply connected case and this is for the compact simply connected case and this really a deep theorem.

So, I will stop with that.

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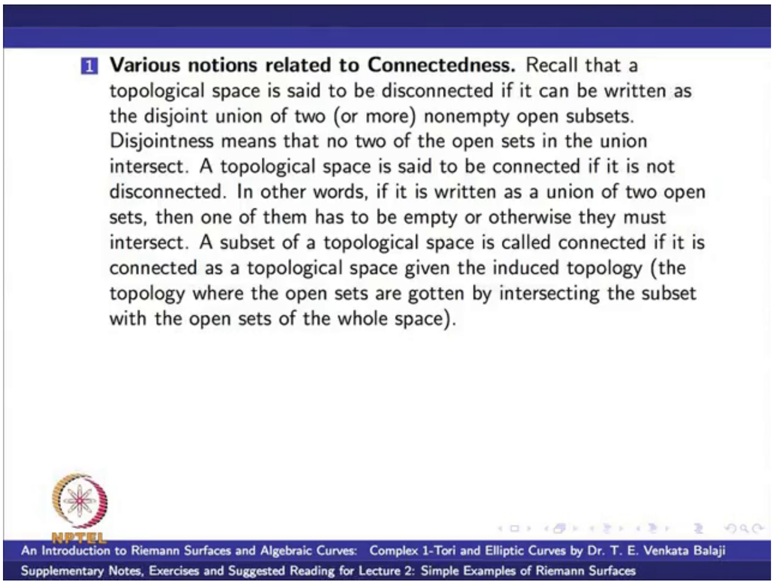

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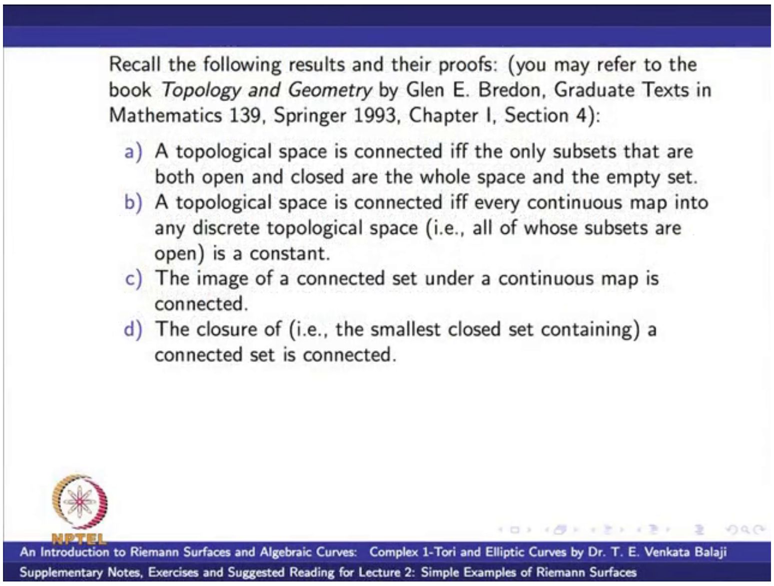

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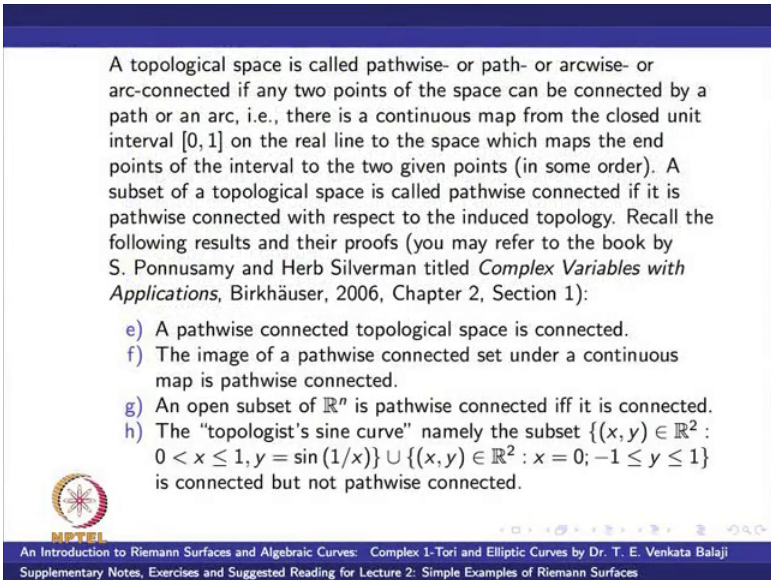

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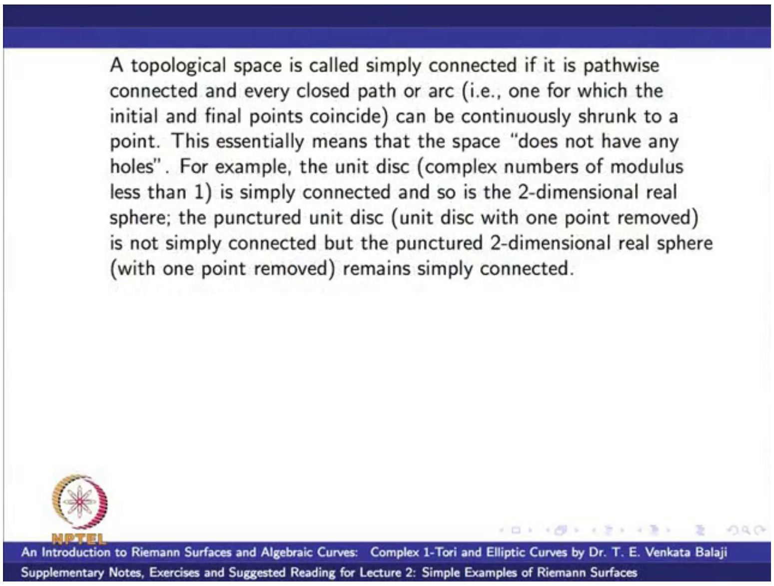

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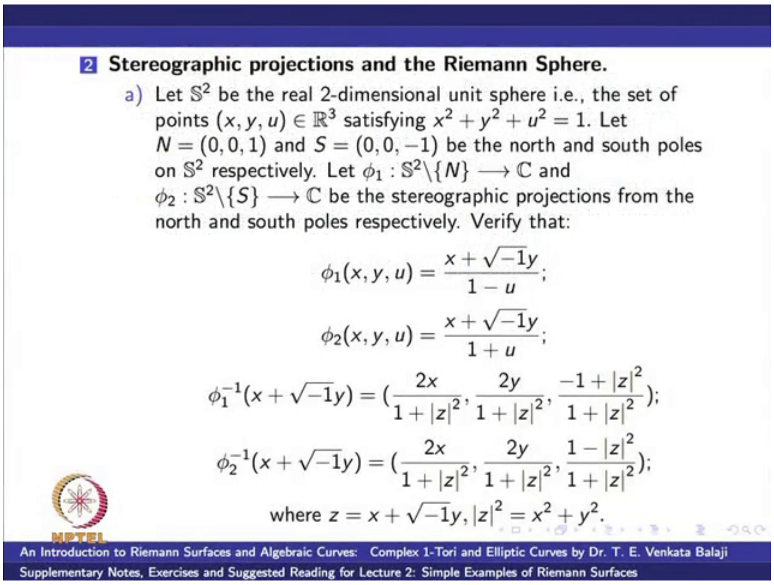

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