An Introduction to Riemann Surfaces and Algebraic Curves
Complex 1 – dimensional Tori and Elliptic Curves
Dr. Thiruvalloor Eesanaipaadi Venkata Balaji
Department of Mathematics
Indian Institute of Technology, Madras

Lecture – 17
Riemann Surfaces with Universal Covering the Plane or the Sphere

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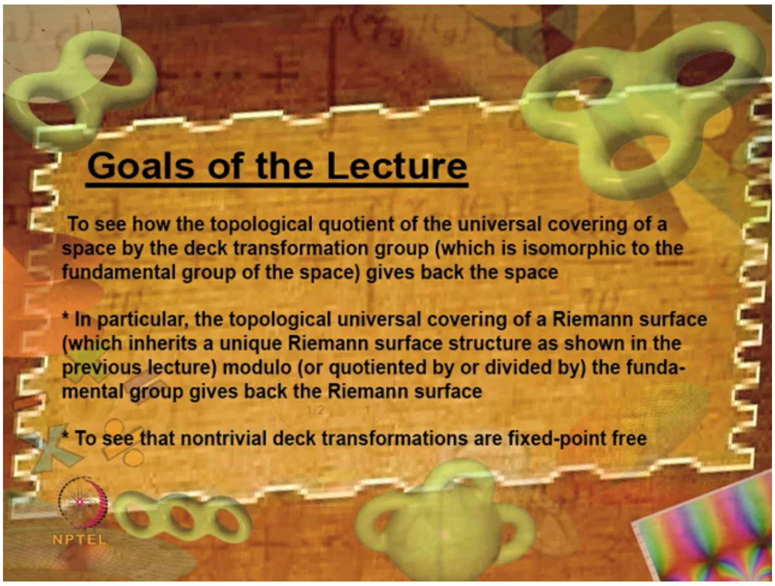

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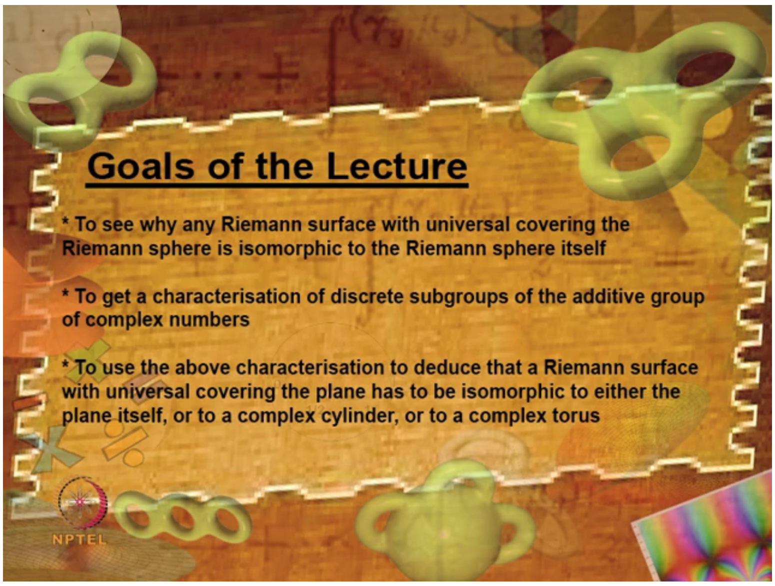

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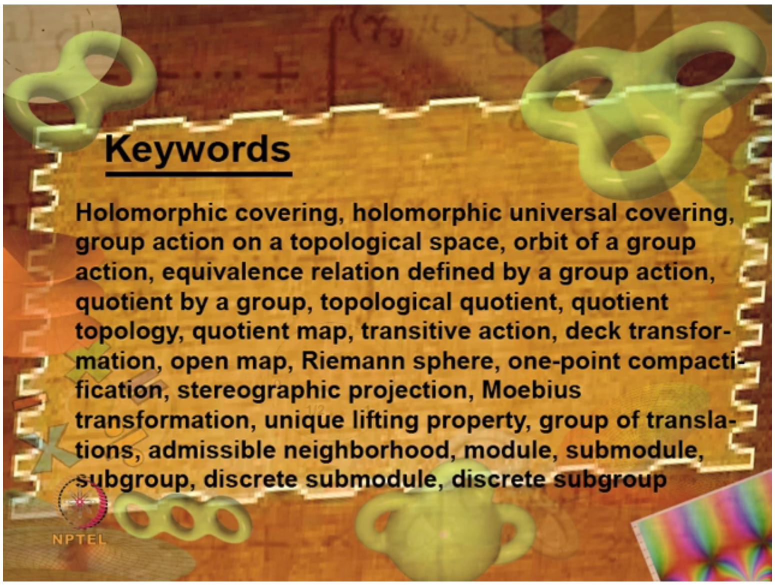

So, let us continue with our discussion. Last time, we saw how the universal covering of a Riemann surface is actually a covering in the holomorphic sense. So, I just want to make another remark in this direction.

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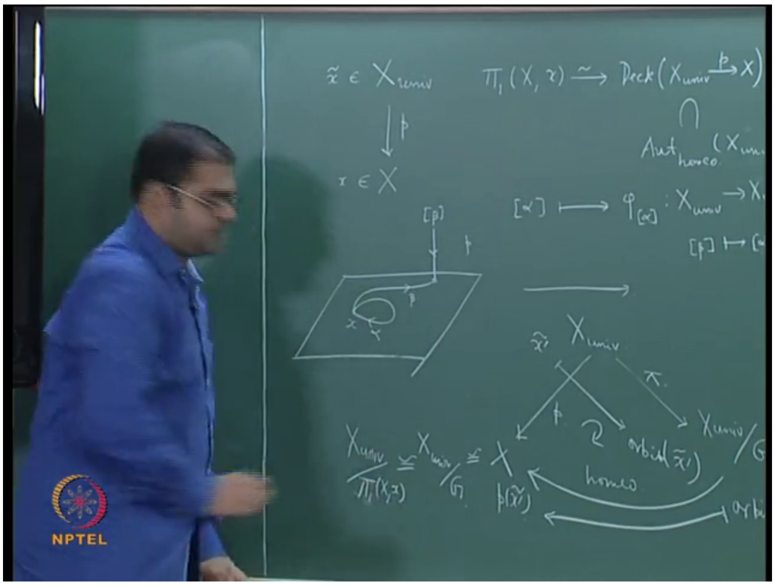

So, let us start with the covering. So, just topological covering, let me say universal covering which we have already constructed for a given topological space, then well you know that if I take this point X then I have this point x tilde above which in our construction corresponded to the constant path at X. So, the way we constructed this universal covering was to consider path starting at x and mod out by fixed end point homotopic line by fixed end point homotopic as the equivalence relation. And, in particular the points above x or just loops at x homotopic class at loops at x and therefore, the fiber over x was just the fundamental group of capital X based at small x.

But, in fact, we had more. In fact, what we had was that the fundamental group of X based at x is identified by an isomorphism of groups with deck transformation group of this covering and this deck transformation group was sub group of the group of automorphisms, homeomorphisms, self homeomorphisms of the universal cover.

So, in fact if you remember the way we did this was we defined the map alpha, suppose, I take an element alpha which is a loop based at x and took it is homotopic class which gives me an element here, then I just sent this to the map phi sub alpha and this map was the map from X sub univ to itself, which would send given any point beta, I would simply send it to alpha beta. So, what this diagrammatically means is that give me a loop give me a path starting at. So, so let me recall that. So, that it is easier for you.

So, here is x and here is my element alpha and there is this path beta and this above this end point beta of one is where the point b given by beta lies. So, this point is mapped to this by p and where does beta go to under this? It goes to again to a path starting from small x and that is just alpha followed by beta and we check that this is an isomorphism of groups and that is how the fundamental group of the base was identified with deck transformations.

And, let me remind you that the deck transformations are all homeomorphisms of the universal covering into itself which respect the projection, which means they will they are homeomorphisms which will act along the fiber directions. A point will under a deck transformation will go to another point which lies in the same fiber as a original point. So, the deck transformation group is acting in the vertical direction.

So, well we have this now what I want to tell you is that I want to kind of give a converse to this in the following sense. So, what I want to say is suppose I start with this universal covering and well I look at the universal covering mod, the deck transformation group, let me call this as G. So, if I go mod g, what is this notation mean? So, what I do is, you see, G is a group and it acts on this topological space universal covering and therefore, this space is broken down into orbits. So, you have an equivalence relation on X sub univ. So, let me write that down.

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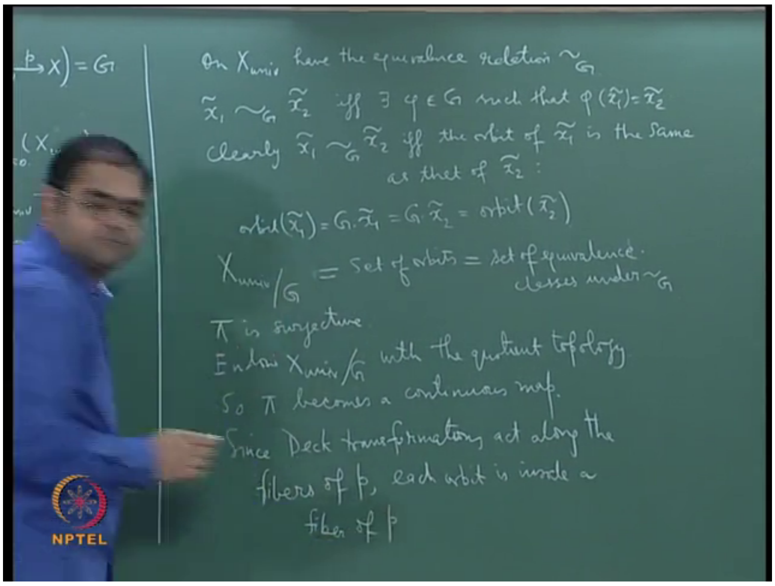

So, on a X sub univ, have the equivalence relation, let me call that tilde sub G which is let me say x 1 tilde is equivalent to x 2 tilde if and only if there exists an element phi in G, such that phi takes x 1 tilde to x 2 tilde.

So, this is just the 2 elements here are considered to be here equivalent to one another. If there is a group element namely a deck transformation which moves this to the other and the point is, this is just saying that this is just trying to break down this space into orbits under the group. So, I am just saying that clearly x 1 tilde is equivalent to G under x 2 tilde, if and only if the orbit of x 1 tilde is the same as that of x 2 tilde. So, what does that mean? That means, orbit of x 1 tilde is just G dot x 1 tilde and this should be the same as G dot x 2 tilde which is the by definition the orbit of x 2 tilde and you know what G dot x 1 tilde means, it means you apply every element of G to x 1 tilde and collect all these elements together that is orbit.

So, and what does this mean? This is just the set of orbits this is precisely the set of orbits, which is the same as the set of equivalence classes under that equivalence relation. So, x sub univ mod G is just by definition this equal to set of orbits that is equal to just the set of equivalence classes under that equivalence relation. It is one and the same thing alright.

Now, what I want you to understand is, so what is this map? This is the let me call this a pi. This is the very natural map which just sends any x prime tilde is just map to orbit of x prime tilde, which is in other words it is map to the equivalence class that it belongs to. So, this is your map alright and it is very clear that this map is surjective by definition because this is just the collection of orbits.

So, pi is surjective, which is obvious and then now what I can do is that since the set on top is a topological space I can put the quotient topology for this set below. So, endow this with the quotient topology. So, which means that you know you put in the minimal number of collection of open sets that makes this map continuous. So, in other words, set here is open, if and only if it is inverse image here is open alright. So, endow the set x univ mod G with the quotient topology then pi also becomes the continuous map because by definition if I take an open set here, I have to check that the inverse image of that set is continuous, but that is how an open set below is defined.

So, this automatically makes pi into continuous map. So, pi becomes a continuous map. Then, I would like to look at these orbits and the point I want to make is that you see if you move the equivalence relation moves an element to another element by a deck transformation, but a deck transformation is supposed to move elements only along the fiber which means all the elements which are equivalent to an given element namely the orbit of the given element has to be only in the fiber over p. So, since deck transformations act along the fibers of p each orbit is inside a fiber of p. Each orbit has to lie in a fiber and fiber of which point you take the point whose orbit you taking, take it is image below and take the fiber over that point, it is lying in that fiber.

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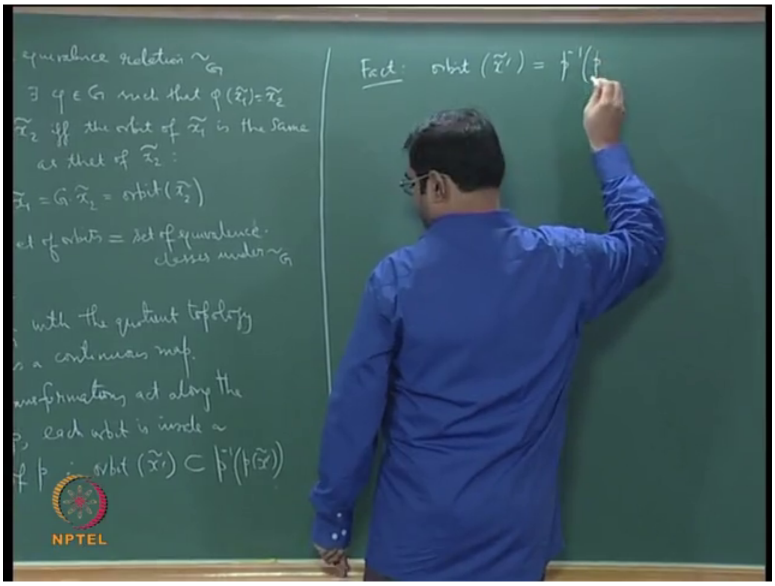

So, I can just say that orbit of x prime tilde is going to be a subset of p inverse of p of x prime tilde. It is going to be in that fiber.

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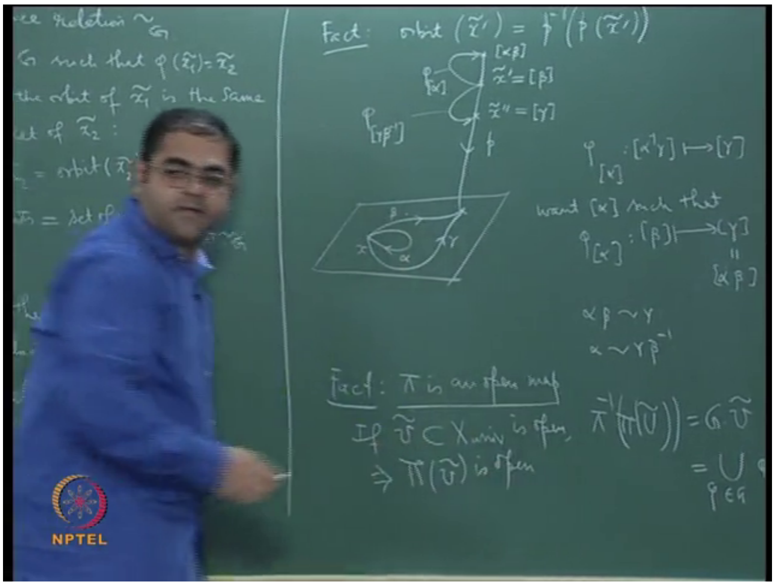

Now, the next thing I want to say is that the orbit is the whole fiber. So, fact is that orbit of x prime tilde is actually equal to p inverse of p of x prime tilde alright and I mean this should be familiar to you if you remember that I told you that the fundamental group below at each point will act transitively on the fiber over that point. I made that remark in an earlier lecture, but even otherwise it is very easy to see.

So, what is happening is you have, if you want you see I have my x alright and suppose I will take a point x prime tilde, then this point is going to be by definition a path beta, because that is what is an element in the universal covering space, the way we have constructed it and this is going to be a path like this. So, it is going to end at a point and this point is a point to which is this is going to be mapped under p. So, this is how my diagram looks like.

And see you know if you give me and how is it that the deck transformation is defined? It is defined in the following way. So, if I start with an element alpha here, as I this is exactly I have written it there, then you see my point to which the deck transformation defined by alpha is phi sub alpha, it maps beta to alpha beta. This is how my deck transformation is defined. And, notice that, this is how it is defined.

I want to say that the orbit is everything. So, what I am saying is let us take another point let us say x double prime tilde and then let me call this as gamma, my claim is that there is a deck transformation that will take x prime to x double prime tilde and what is it, you can guess. I will just have to apply; I will just have to take the point alpha inverse gamma. So, you can see that phi sub alpha will take alpha inverse gamma. So, phi sub alpha will take alpha inverse gamma, but I want beta to let me write this down, it will take alpha inverse gamma to gamma that is not what I want. I want beta to go to gamma.

So, essentially what I need to do is, so, in fact, I am searching for a suitable alpha alright I am searching for a suitable alpha such that, it will take. So, want an alpha such that phi sub alpha takes beta to gamma. The moment I get such an alpha, then this is a deck transformation that will move the point beta to the point gamma; this is what I want alright. So, if you write it down what it will tell you is that that this gamma has to be alpha, it has to be alpha beta by definition because p alpha takes beta to alpha beta.

So, you can solve for alpha from this. So, you will find that alpha is just alpha beta is homotopic to gamma. So, you will get alpha is homo topic to gamma beta inverse. So, I will just have to take. So, you see this gamma is, mind you, gamma is some other path in x, which also ends at the same point as beta ends because both of them are points above this end point. So, gamma followed by beta inverse is a loop based at x and if I take that loop as an alpha then that phi sub alpha, the corresponding deck transformation will move beta to gamma. So, it is very clear therefore, that, in other words if I want to move from here to here then what I have to use is phi sub gamma beta inverse, this will do the job. So, what I proved is, give me any other point x double prime tilde I can move x prime tilde to x double prime tilde by a deck transformation which means that the orbit will contain all points in the fiber. So, that establishes this.

So, if you now go back to this diagram and suppose, I put also this original covering into the picture then what you see is that I also have I can also define a map like this I can define a map like this, namely I will take on the orbit of a certain x prime tilde and I send it to p of x prime tilde and you can see that this is a bijective map.

So, in other words this set is just the orbits, but the orbits are precisely the fibers and therefore, your this set is just taking all these fibers here and thinking of each fiber as a point. So, you get a map like this and this map is bijective you can easily see. And you can also see that you know that this pi will be actually an open map and why is that so; so, let me write pi is an open map that is very easy because, you see you know if I take an open set here, then suppose I call it as U tilde, if U tilde in x sub univ is open then I have to check that pi of U tilde is open to say that pi is an open map I have to say it takes open sets to open sets.

So, I have check pi of u tilde is open, but then because of the quotient topology this is the same as trying to check that pi inverse of pi of u tilde is open. So, what is pi inverse of pi of U tilde? Pi inverse of pi of U tilde is simply G dot U tilde.

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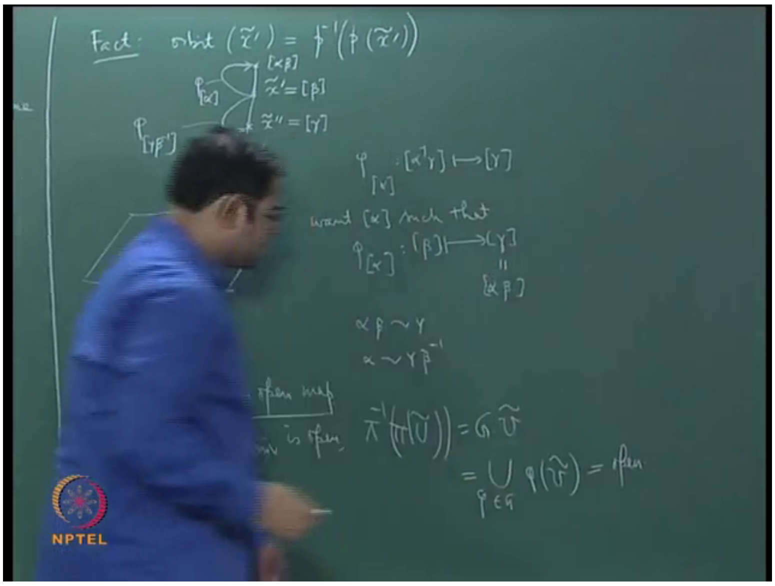

See, when you take the inverse image of a point here you get the whole orbit. If you take a set above and you go down and then you take the inverse image you are only taking all possible translates of that set by elements of G, because look at what is happening point wise. You take a point here, you go down, you take it is image that will be the orbit of that point if you take the inverse image you get the full orbit of that point and what is the orbit of the point? It is the translate of that point by the whole group G, which is the deck transformation group. They say what happens to a point also happens to a set. If I take a set here, I push it down and then if I take it back I will simply get G dot that set which means all translates of that set.

So, this is just all translates of U tilde, but these are all translates of U tilde by elements of G and which are any way homeomorphisms. So, this is a union of open sets. So, it is open. So, this is see this is just equal to union G belonging to G, if you want let me call this phi belonging to G, phi of U tilde. This is what this is and you see each phi is a deck transformation. So, it is a homeomorphism of the universal covering. So, phi of U tilde will be an open set isomorphic to U tilde and this is union of such open sets and therefore, this is open. So, this calculation will tell you that pi of U tilde is open. So, this will tell you that pi of U tilde is open.

Now, let us go back to this diagram. This is an open surjective map and of course, this is an open surjective map because this is the covering map and this map here is bijective and it is actually a homeomorphism, why? Because, you see it is bijective, it continuous if you take an open set here you pull it back I will get an open set and I will get an image here I will get an open set because this is open. So, the inverse image of an open set is open. So, it is continuous and I can also argue the other way round. If I take this map will also be an open map because, if I take an open set here I take it is inverse image that will be an open set and then if I take the image this is an open map. So, it is image is open. So, this is an open continuous bijective map, so, it is a homeomorphism.

Therefore, in other words, this is none other than that. They are homeomorphic and under this homeomorphism you can identify p and pi. If you identify every point below with the orbit above as a set then these 2 can be identified and this p is just the natural map sending an element to it is orbit and what is the moral of the story the moral of the story is therefore, that if you have a universal covering then X can be written as X universal modulo deck transformation group.

So, let me write that down. So, let me write this here, this is a homeomorphism and this diagram commutes that is this followed by this is this and this followed by it is inverse is that and X is therefore, identified by this homeomorphism with X the universal cover, modulo the deck transformation group. So, the reason why I am doing this is the following I am just trying to tell you is that what I am trying to tell you is that if you take this universal covering, not only is the fundamental group below identified as a sub group of automorphisms above, but the top space modulo that subgroup of automorphisms which is actually the fundamental group below is actually the space below. This is actually a quotient of that by the deck transformation group by the fundamental group.

So, I can also write it as X universal mod pi 1 of X comma x, I can also write it like this, because, after all deck transformation group is isomorphic to the fundamental group. So, you must always remember this when you have universal covering the fundamental group below is a sub group of the automorphism group above and there is space above, modulo the fundamental group start of as going modulo deck transformation group will give you exactly the space below and this covering projection is actually the map that sends every element to it is orbit, that is all. So, that is how you should think of it.

Now, let me add. So, all this is in the topological category, but now assume that X is a Riemann surface. Then I have told you already that this universal covering gets canonically

Riemann surface structure. A unique Riemann surface structure such that p becomes the holomorphic map. So, what happens is that the Riemann surface structure can be transported from here to there, but then because this pi is after all p which is a local homeomorphism after this identification I can again take this Riemann surface structure and push it to below. So, I will get Riemann surface structure here and that Riemann surface structure is nothing, but this Riemann surface structure borrowed by this homeomorphism.

So, whatever I have said will also work in the holomorphic category. If X is a Riemann surface and X universal is the universal cover of the Riemann surface which gets the unique Riemann surface structure then not only is the fundamental group below identified as a subgroup of the holomorphic automorphisms of the universal cover; namely, the deck transformation groups are now holomorphic maps, holomorphic automorphisms, but if you take this Riemann surface above and go modulo the fundamental group below which is now thought of a holomorphic automorphisms, a sub group of holomorphic automorphisms the quotient you get is again the same Riemann surface. You get back the same Riemann surface and everything is valid in the holomorphic category. So, this is something that I want you to bear in mind.

Let's proceed. So, what I am going to do next? We need to start applying all this theory to get it hold of you know first classification of Riemann surfaces. So, let us first look at Riemann surfaces which admit the Riemann sphere as the universal cover, as even as the cover.

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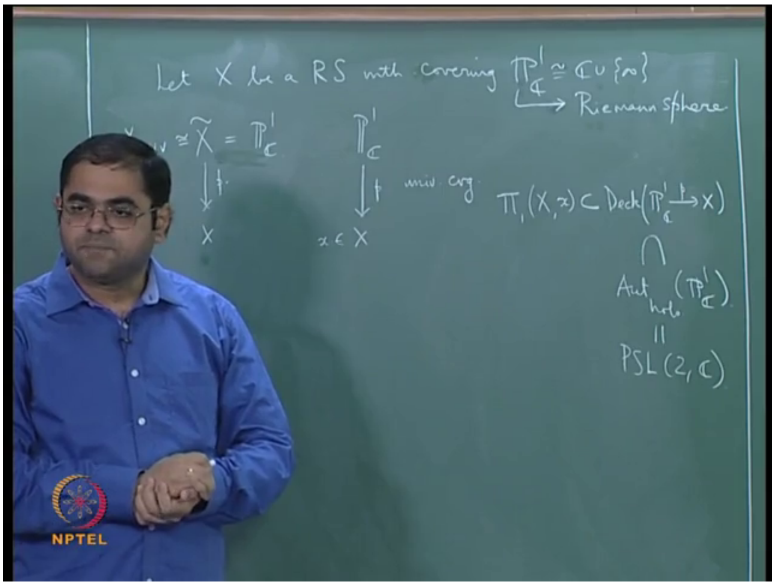

So, let x be Riemann surface with covering P 1 C which is just C union infinity, which is which is a Riemann sphere. So, if you remember P 1 C being the Riemann sphere is the same as the one point compactification of the complex plain by this stereographic projection and you know that we made this into a Riemann surface in one of our earlier examples and that gives us the Riemann surface given by the Riemann sphere. So, the suppose X is a Riemann surface with covering this.

So, I have the following situation. So, here is a universal. So, I have X tilde I have a covering and I know this is P 1 C. Of course, I can write isomorphism up to isomorphism, but I do not worry about that because if there is an isomorphism I will just compose and that will also be a covering map. So, now notice that P 1 is a Riemann surface structure on the 2 sphere which is simply connected.

Therefore, this has to be the universal cover. So, this has to be X sub univ up to isomorphism because I told you that you take any covering with the covering space above being simply connected then had if the head it has to be the universal covering and then you give any 2 universal coverings, they are unique up to unique isomorphism and that isomorphism is fixed, if you fix a point here and it is imaging there alright. So, this is the universal covering. So, my situation is that I am just having P 1 C to X by this projection and this is universal covering.

Now, what do we know? We know that you see if I fix a point x here and if I fix a point x tilde above then, I do not need to fix a point above, basically, we know that the fundamental group of capital X small x is identified with the deck transformation group of this cover; this is what we have seen and this is the subgroup of the automorphisms, now holomorphic automorphisms of P 1 C.

So, the deck transformation group is now a sub group of holomorphic automorphisms of the universal cover and what are the holomorphic automorphisms of the universal cover? You know, that this is just full group of mobius transformations, which is given by PSL 2, C. So, every automorphism of the extended complex plain, every holomorphic automorphism of the extended complex plain has be a mobius transformation and if you write the mobius transformation as the z going to a, z plus b by c, z plus d then you get this matrix a b c d and then you normalize this matrix by requiring that a d minus b c is equal to 1 and then of course, you have to go mod signs and therefore, you get this identification alright.

Now, I want you to understand the following thing. See, the first statement I want to make is a deck transformation. So, I want to say 2 deck transformations, if they are same if they agree at one point, then they are the same everywhere.

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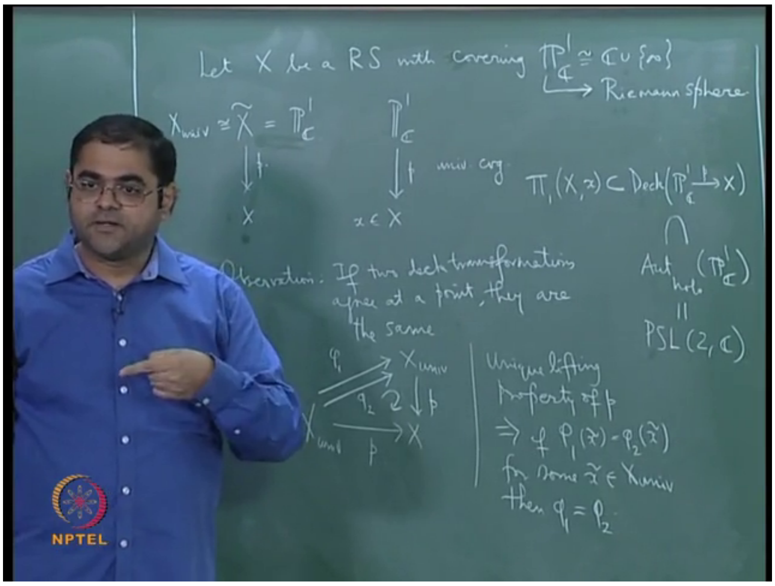

So, let me make a statement, observation; if 2 deck transformations agree at a point they are the same. Why is this true? This is because; they are both lifts of the projection, of the covering projection. They are all lift and you know there is a unique lifting property. So, what is happening is that I have X sub univ and here is my X and here is my covering projection and I write that again here and suppose I have phi 1 and phi 2 are 2 deck, which are you know automorphisms from universal covering to itself; then both of them are lifts of p, phi 1 followed by p is also p, phi 2 followed by p is also p. I mean that is exactly this statement which says that the deck transformations move points along the fiber.

So, that is how the deck transformation group is defined. So, it means this followed by this is this, but then I told you that a covering space a covering map has a unique lifting property that is if you have a lifting, if you have 2 lifting's, if they agree at a point then they agree everywhere. So, what this will tell you is that unique lifting property of p will imply that if phi 1 of x tilde is equal to phi 2 of x tilde for some you know x tilde in the universal cover then phi 1 is equal to phi 2. So, this is just the uniqueness of lifting that I am using here.

So, if 2 deck transformations agree at a point they are the same. So, what I wanted to tell you is that I want to say that from this I want to deduce that no deck transformation can have a fixed point. No non trivial deck transformation can have a fixed point. Why, because if a deck transformation has a fixed point then it agrees with the identity map which is also a deck transformation at that point. So, it has to be identity everywhere by this. So, the upshot of this is no deck transformation different from the identity has a fixed point alright.

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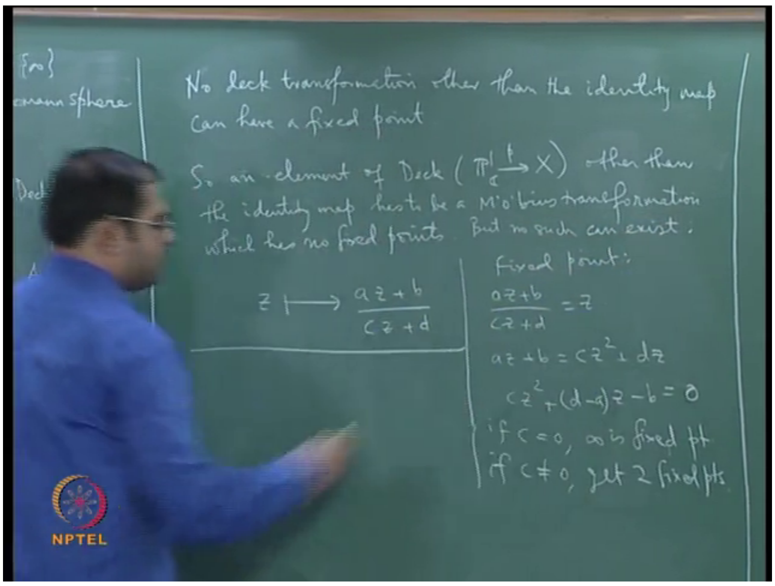

So, let me write that down, no deck transformation other than the identity map can have a fixed. No deck transformation other the identity map can have the fixed point now. So, you see pi 1 X the fundamental group. In fact, I should write isomorphic to this is not this is not a sub I am sorry please correct it this should be isomorphic to and this is sub here you see therefore, what you are looking at are mobius transformations different from the identity and which have no fixed points. If you are looking at non trivial elements here corresponds to mobius transformation. It corresponds to mobius transformation which has no fixed points and such a mobius transformation cannot be existed. So, the conclusion is that this is the trivial group.

So, an element of deck P 1 C to X, p has to be other than the identity has to be mobius transformation which has no fixed points, but there is no such transformation, why? Because, you see a mobius transformation looks like z going to az plus b by cz plus d. Now, you see if I try to solve for fixed points then you know I will get I am just solving az plus b by cz plus b is equal to z, I am just solving this equation alright and this amounts to solving az plus b is equal to cz square plus dz. So, what I get is, well, I get cz square plus d minus a times z minus b equal to 0.

Now, notice that this is the quadratic equation see if c is 0, this is not a quadratic equation and if you see if c is 0 and a is not equal to d, then I get a fixed point and if c is not 0, I have to 2 solutions to the quadratic equation which will give fixed points. So, the moral of the story is that I will at least get I will get a fixed point, if it is not the trivial transformation, it is not the identity transformation.

So, this will tell you that there is no element of the deck transformation group other than the identity. In fact, what I want to say is that if c is 0, if you are wondering for a moment; if c is 0, then infinity becomes a fixed point. Mind you, we are looking the at the point infinity also as a point, also a point here the Riemann sphere. If c is 0, infinity is a fixed point; if c is not 0, you get 2 fixed points. Of course, the 2 points may be the same you count them up to multiplicity. So, the moral of the story is that you cannot have any non trivial elements in this deck transformation group; that means is a trivial group and that means, the fundamental group of X is trivial.

If the fundamental group of X is trivial, then X is itself simply connected and the identity map from X to X is itself is a universal covering. Because, identity map of a space into itself satisfies all the properties of a cover of a covering and if the space is simply connected that also is the universal covering and I told you the universal coverings are all isomorphic.

Therefore, what this will tell you is that you see it will tell you that this map P, X has to be identified with X must be the same up to holomorphic isomorphism as a Riemann sphere and this is map P will just have to be an automorphism of the Riemann sphere and therefore, it has to be an element of PSL 2, C. So, this discussion tells you that any Riemann surface which has P 1 as a universal cover has to be isomorphic to P 1 itself and nothing more.

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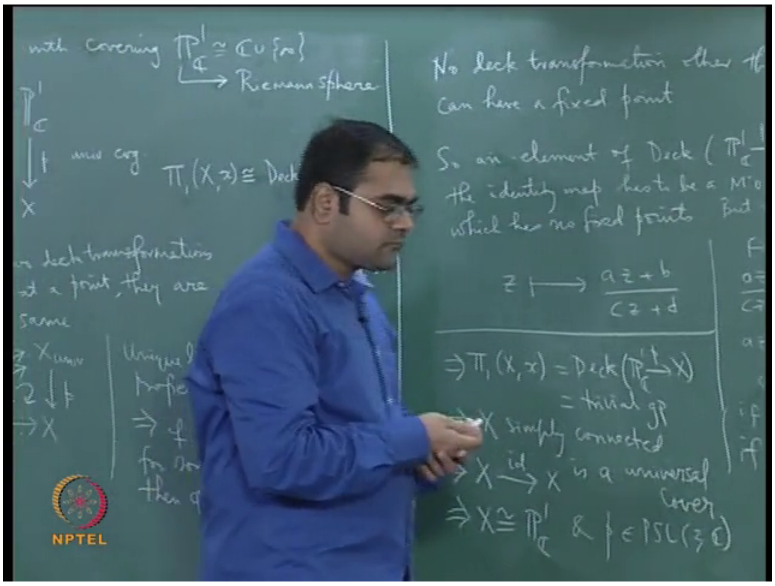

So, let me write that down here. So, let me add here, but no such can exists and this is the. So, what this implies is that pi 1 X comma x is equal to the deck transformation group of P 1 C, p X trivial group; namely, it consists of only the identity element and this implies X is simply connected, but this implies that this X to X identity map is a universal cover and that will imply that X has to be isomorphic to P 1 and once you make this identification of X with P 1 this map P has to be just an automorphism of P 1, so, it has to be just an element of PSL 2, C and so, this is the proof of the theorem that I stated earlier that any Riemann surface which has P 1 as a covering has to be just simply P 1 it iself. So, you do not get anything more right.

So, let me go to the next case namely, I am going to look at Riemann surfaces which have universal cover the complex plane. So, you know the universal cover for a Riemann surface has to be a simply connected Riemann surface and the fundamental uniformization theorem tells you that the there are only 3 such; namely, a simply connected Riemann surface is either Riemann sphere or it is a core complex plain or it is unit disk which is a same as upper plain alright. So, I have covered the first case then the universal cover is the Riemann sphere; now, I am going to look at the case when the universal cover is the complex plane. So, let us go to that.

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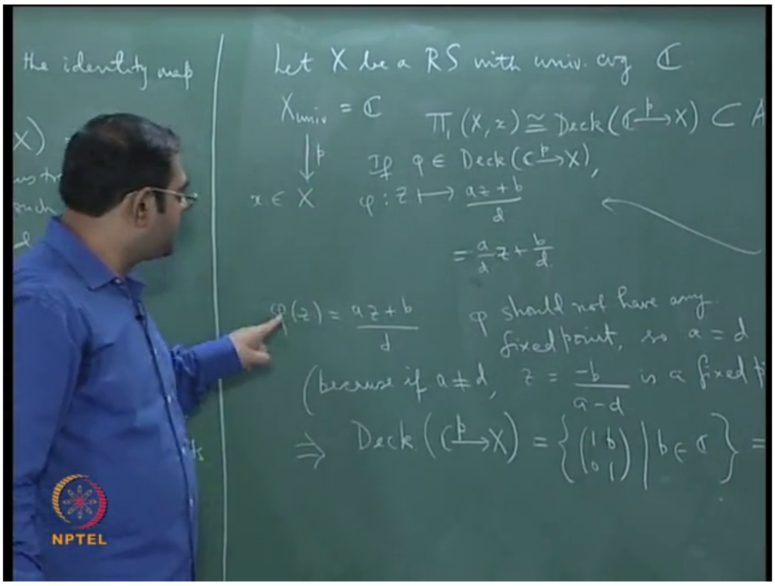

So, next, we will look at this let X the a Riemann surface with universal covering C. So, I am going next look at Riemann surfaces universal covering the complex plain. So, the situation is like this you have X sub univ you see and this is the covering map and here is my X. Again we have if I fix a point x below in small x and capital X then we have again an identification of the fundamental group of X based at x with the deck transformation group of this covering which is C to X and this is sub of the holomorphic automorphisms of C, which is the space above. Now, the holomorphic automophisms have c or if you check they are going to be again mobius transformations.

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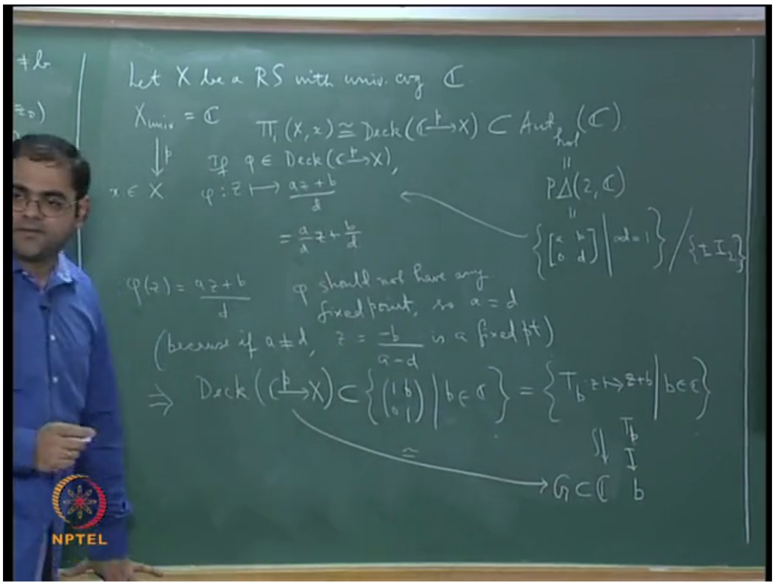

So, one writes them as P delta 2 comma C and these mobius transformations are precisely all those writable in this form a b let me say 0 c, a b c d such that a d equal to 1 and modulo plus or minus the identity matrix. See a general mobius transformation is written as a z plus b by c z plus d and it is identified with the matrix a b c d and the condition you put is a d minus b c is equal to 1 if you take a general mobius transformation. But, here I want automorphisms of C and the automorphisms of C will have this you know upper triangular form. So, you know if it is an automorphism of C infinity has to be fixed. Therefore, I cannot have this C term here. So, I have this upper triangular.

Now, as I have already pointed out if C is 0, then infinity is a fixed point and that is exactly what happens. Now, what are these, let us look at all these mobius transformations. See, if you take an element from here it is going to look like z going to well a z plus b by d and well I can as well write it as you know a by d z plus b by d, I can write it like this because I can just divide throughout by d and again as I explained here if you look at the observation that a deck transformation has to be deck transformation different from the identity has to be one that has no fixed points. So, if you put that condition here you will see that this coefficient has to become 1.

So, let me write that down. So, if phi is a deck transformation and phi takes z to this. So, you see phi of z is just a z plus b by d alright and the condition on phi is that it should not have any fixed point, phi should not have any fixed point. So, that will happen only when the coefficient of z is 1. I think that is quiet obvious I think. So, how do I write that down, I just have to say that. So, you see, a equal to d, why? Because, if a is not equal to d I can get a fixed point z is equal to minus of d by, I am just solving a z plus b by d is equal to z. So, I will get a z plus b is equal to d z. So, I will get z e z minus d is equal to minus b. So, I will get z is equal to minus b by a minus b. So, this will be is a fixed point. So, yeah a z plus b is equal to d z. So, a has to be equal to d and, but a d is 1. So, that means, a square has to be 1, a has to be plus or minus 1, but I am doing modulo plus or minus identity for the sign.

So, I can take a is equal to 1. So, the up short of all this is that is deck transformation group can be simply identified with matrices of this form 1 0 1 a if I use let me continue using b such that b is a complex number. So, the deck transformation group it can be identified in this form and what are these maps, is a just translations these maps these are just the set of all p sub which is the map that sent z to z plus b, where b is a complex number.

You see when a is equal to d equal to 1, this map just becomes z going to z plus b, which is translation by b. So, the deck transformation group is I should say is identified with a sub group of, I should be careful deck transformation group, is identified with the sub group of this which is of this form I cannot say that everything is here right, but this is a sub group of this right and mind you this can be, so, this group of translations if you look at it group under composition can be identified with simply with the complex numbers as a group under addition.

So, this last group is just isomorphic to the complex numbers we by just sending T sub b to b and you know if you compose T sub b 1 and T sub b 2, it will translation by b 1 plus b 2 and the composition will go to b 1 plus b 2. So, this is the group homeomorphism. So, the moral of the story is that the deck transformation group which is isomorphic to the fundamental group becomes as a group first of all it becomes Abelian because it is a sub group of these group of this group is a group of translations, that is, a Abelian. So, the first thing that you infer is the fundamental group has to be Abelian, that information you get. Then, the second thing is you also realize that the deck transformation group is an additive sub group can be identified with additive subgroup of the complex numbers under addition. So, let me write those remarks down.

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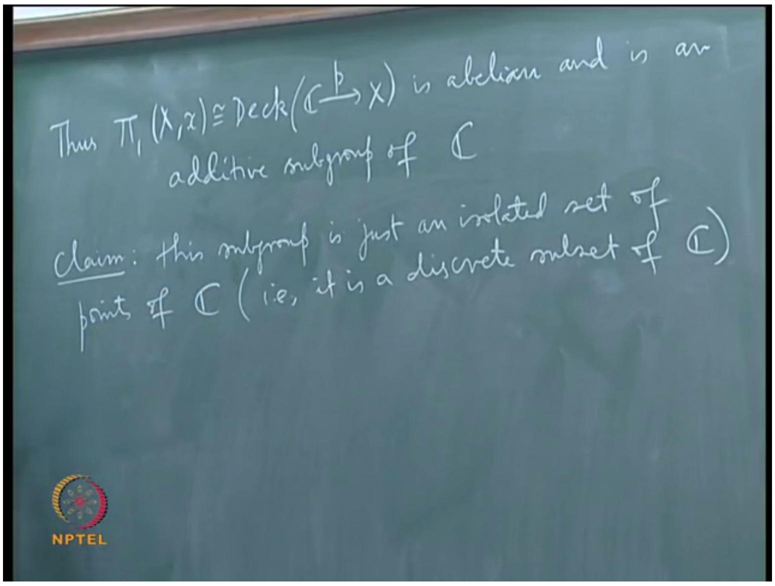

Thus pi 1 X comma x is Abelian. So, that is the information you get and is an additive sub group subgroup of C it is an additive subgroup of C. Now, there is one more part to this that one has to understand before you proceed and that is to understand what this is topologically as a subset of C. So, what I am now going to tell you is a topological as a subset of C it is discrete. It consists of only an isolated point that is what I am going to say next. So, claim; this sub group is just an isolated set of points of C, that is, it is a discrete subset of C.

So, it is a discrete subset of C, that means, every point of that sub group if you identify to the with the point of C then it is isolated away from the points and it is a discrete subset means there is no accumulation points to this set of points. So in fact, what I mean here is the subset of these the subset of points given by this subgroup is actually a discrete set. So, this is a very important and one has to prove that and this is how one proves it. So, let us do deduct sure add absurdum let us assume that this is not the case.

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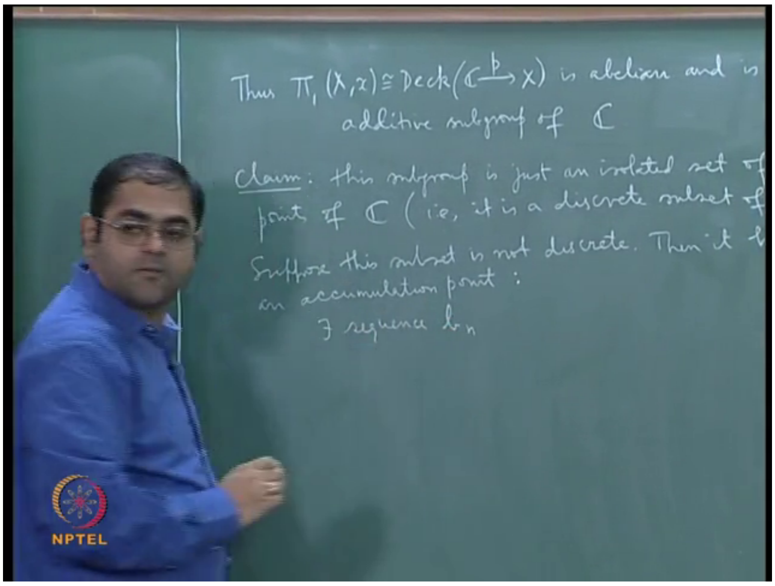

Suppose, this subset is not discrete, that means, it will have an accumulation point. It will have an accumulation point and then you can pick a sequence of points here which are distinct points and which will converged to that accumulation point, then it has an accumulation point.

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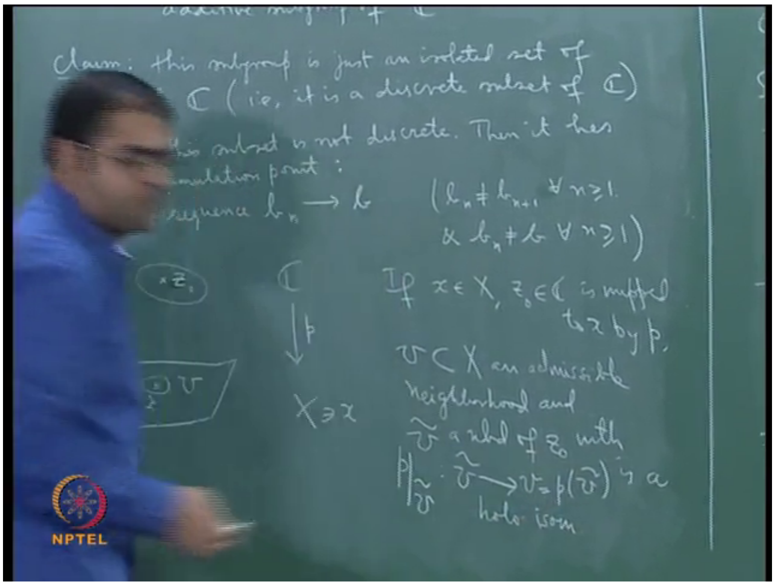

So, there exists a sequence b n, which tends to b and each b n is difference from b n plus one and they are all different for all n and of course, each b n is different from b. This is what accumulation point is, you have to have sequence of distinct points which go to a limit which is also different from each one of this points. So, there is an accumulation point. So, what this means is that you see, this is going to create problems because and that is what is going to give as a contradiction.

Now, you see my situation is as follows, I have C here I have p, I have my X below and let me draw a diagram. You see, suppose, I fix a point small x in capital X, then you know that this is a covering. So, this point will have an admissible neighborhood. So, suppose I take this admissible neighborhood U here, then it is inverse image under p will be a disjoint union of open sets, each of which is mapped by p homeomorphically onto X. So, in other words you know if I take a point x tilde here.

So, since this is the complex plain I will take the take the point as Z, if you want that I will take the point as say Z naught. So, if I take a point Z naught here then I will get neighborhood surrounding Z naught which will be just I can call this as a if you want I will call this as U tilde and p recitative to U tilde from U tilde to U will be a homeomorphism that is because of the covering property of the covering space. So, if x belongs to X, Z in Z naught in Z is mapped to x by p, U in X an admissible neighborhood and U tilde neighborhood of Z naught with p recitative to U tilde from U tilde to U, U is just p of U tilde is a holomorphic isomorphism.

So, all this happens because this is a covering space. Now, you see you take this neighborhood which contains Z naught and apply the translation by b n, apply translation by any b n.

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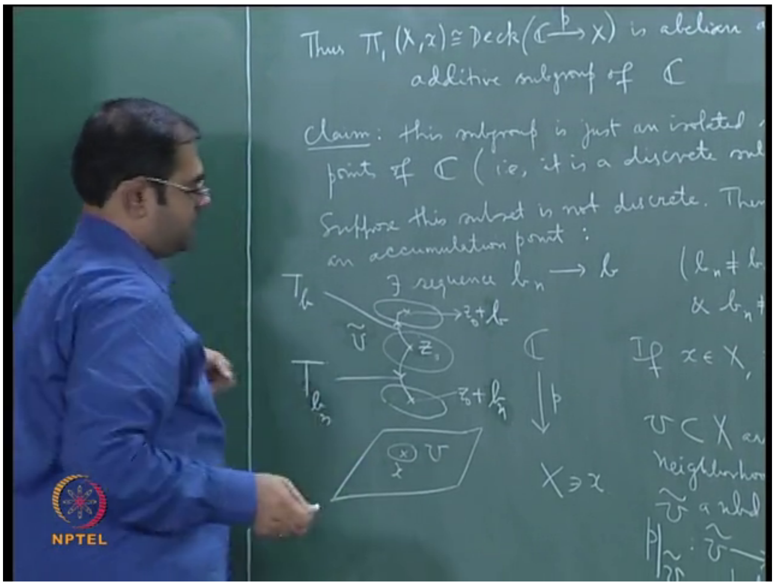

Then what will happen, if I apply translation by b n, I will get another neighborhood at and this will now be and neighborhood at Z naught plus b n. So, this will be you see this point will be Z naught plus b n and this is actually action by the deck transformation of element which is given by translation by b n and mind you translation by b n has been identified with the point b n with C alright. So, this is how translation by b n will act and notice that these 2 will be disjoint. These 2 will be disjoint because translation by b n is anyway holomorphic automorphism and the translation by b n is a deck transformation. So, it will only move along the fiber direction.

So, it will move one neighborhood like this into another and these 2 will be disjoint, because both of them are inverse images of under p of this U and in the same way you also have translation by b. So, here I have one more here this point is now going to be z not plus b. B is this b. So, I am using different notations for b, let me write the same b here. So, this is Z naught plus b and I will get another neighborhood. And, so long as b is not 0, these 2 are different if b is 0 then these 2 will be the same, but nevertheless these 2 are different in any case these are always different because b n is not equal to b ok.

Now, you see you have an inherent contradiction. The contradiction is see b n tends to b therefore, the distance between b n and b can be made as small as I want; that means, the distance between Z naught plus b and Z naught plus b n can be made as small as I want, but you see these 2 are disjoint. So, it is not possible to do that if I choose n sufficiently large I can make the distance between Z naught plus b and Z naught plus b n as small as I want and that is because I can make b n minus b as small as I want and that is because b n tends to b. But, if I make the distance between these 2 points as small as I want then these 2 have to intersect because they are all in the complex plane, but then they cannot intersect because they are all various sheets over this admissible neighborhood they cannot intersect. So, this contradiction will tell you that your assumption that there is an accumulation point is wrong. So, let me write that down. So, the moral of the story therefore, is that this subset of points of C is actually discrete subset. So, let me write that down here.

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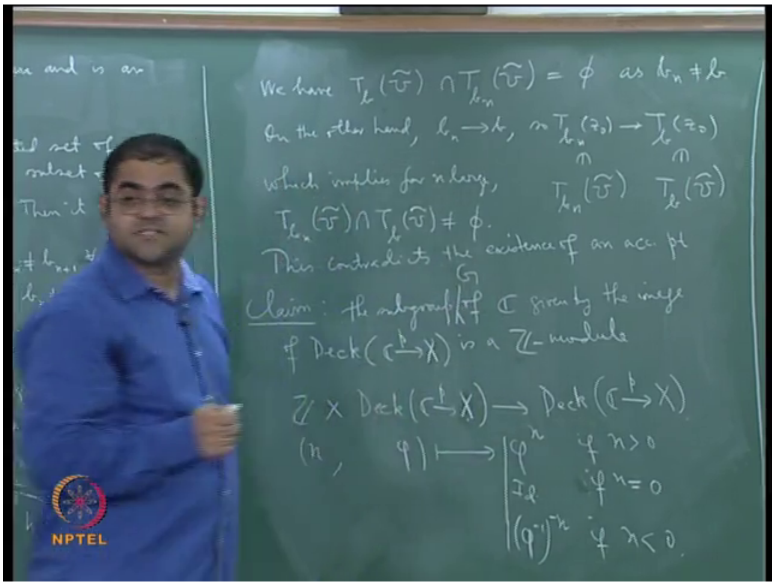

So, that this is. So, this is translation by b and this is translational by b sub n so. So, let me write that down. We have translation by b of U tilde intersection translation by b n of U tilde is always empty because they are translation of b n by U tilde and translation of b by U tilde they are always going to be distinct because b n is not equal to b, as b n not equal to b. On the other hand, b n tends to b. So, translation by b n of Z naught tends to translation by of Z naught and translation by b n of Z naught is going to belong to translation by b n of U tilde and this guy is going to belong to translation by b of U tilde.

So, these 2 sets have to intersect, if b n tends to b, but that is not possible which is not possible which implies for n large, translation by b n of U tilde and translation of b by U tilde has to be non empty, this is the contradiction. This translation by b n of U tilde and translation of b by U tilde has to be non empty. So, this is contradiction the existence of an accumulation point. So, therefore, the claim is proved.

So, you know your fundamental group is Abelian. It is identified with the sub group of additive group of the complex numbers that sub group has a subset it discrete now there is one more point, which is a last point I need to complete the analysis. The last point is that not only is this subgroup of C is actually a Z module. So, that is an input from algebra. So, claim subgroup of C given by the image of the deck transformation group is a Z module. So, this is the next point.

Now, so, I need to tell you what a module is. So, a module is to a ring, what a vector space is to a field, a module is to a ring as to vector has to what a vector space is to a field. So, what is a vector space over a field you have the vector space is basically an Abelian group and they will then there is something called scalar multiplication that allows you to multiply elements of the field which are called as scalars with elements of the vector space and this behaves well with respect to addition and so on. If you use the same rules you can define the notion of a module over a ring and the point is that this subgroup becomes the Z module alright.

So, you see and it is also very clear that is because you see I have to take an element of Z namely an integer I have to multiply an element of the subgroup and produce another element and that is very clear if I take an integer n and multiply by a translation it is just that translation by n times if n is positive if it is the inverse translation minus n times, if n is negative and it is identity if n is 0, that is very obvious way of defining this.

So, let write that down. So, Z cross the deck C p sorry this is not p this is X to deck. You can also look at it in this way you take an integer n and you take deck transformation you know I just send it to phi to the power of n which is phi composite p composed with phi n times if n is positive and I send it to identity if n 0 and I send it to phi inverse power minus n if n is negative and this is how the deck transformation groups becomes the Z module and if you want to identify it with translations then you know how it is going to be. So, fine it is a Z module.

So, for convenience let me do the following thing. So, let me call this sub group G let me call it something d or even let me call it as G; then, namely, what I am saying is that the image of this that is so this is G here inside C. It is an additive subgroup of C and this is just the isomorphism of this, just the image is here and what I have shown here is first that the G is additive subgroup of C which is as a subset it is discrete and here I have G is also a Z module. So, the up short is G is a discrete sub module, Z sub module of the additive sub group of C, module as a Z module. Now, I am going to use this.

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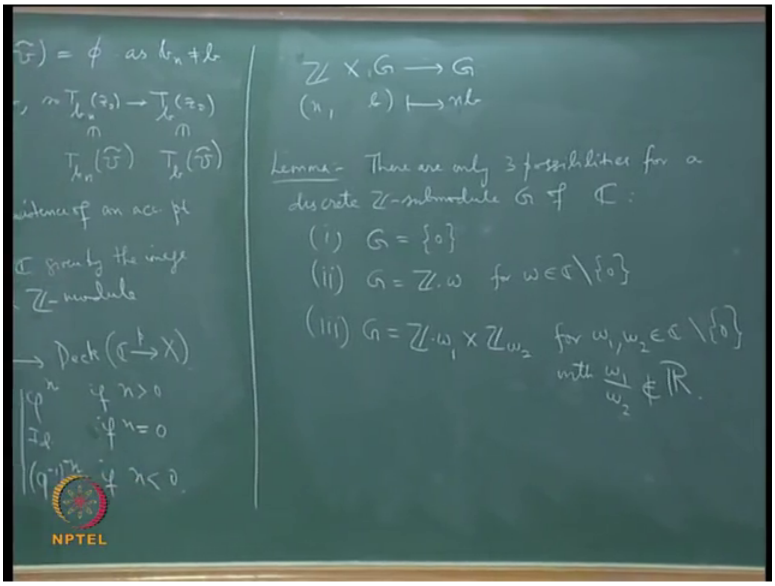

So, before that let me write this I have Z cross G to G if I write it in terms of G which is the image of this and under the isomorphism the identification with of a translation with the element by which you are translating then you know this is just going to be n comma if I take b it is simply going to be n times b this is a natural module structure and this b is to be thought of a translation by b this is translation by n times b. So, this makes G into Z module.

Now, I am going to make another claim and this claim is very important. So, here is the small. So, maybe I can call it lemma there are only 3 possibilities for a discrete Z sub module G of C. So, my G is of course, is discrete as a sub set of C and it is Z sub module right and there is this lemma which is there are 3 possibilities; number 1: G is just a 0. So, G is trivial. Number 2: so, when I say G is trivial it means that it contains only translation by 0, which is just the identity, so it contains only the identity map. In the second thing G is Z dot omega for omega which is a nonzero complex number. So, G is just integer multiplication, it is just translation by integer multiple of a fixed nonzero complex number and the third one is G is translation by 2 nonzero complex numbers omega 1 and omega, with omega 1 by omega 2 not the real number. These are the only 3 possibilities.

Either G is just 0 or it is integer multiples of a single nonzero complex number or it is linear integer combination of 2 nonzero complex numbers with non real ratio and, but you see this G is supposed to be the deck transformation group and that is pi 1. So, you see what it means it means there are only 3 possibilities for pi 1 of x. So, pi 1 of x is either 0 or pi 1 of x is isomorphic to Z or pi 1 x is isomorphic to Z cross G and you also know that the covering can the space below can be written as this module of the deck transformation group.

So, in the first case, X is C mod 0 which is just C. The second case, it is C modulo translates by a integer translate translates by an integer multiples of 0 of a nonzero complex number which you know will give you Riemann surface structure on a cylinder and the third one will give you a Riemann surface structure on torus. So, if one proves this lemma one gets this beautiful theorem that you know whenever a Riemann surface has universal covering C then the X has to be either C itself which is the case when G is 0 or it has to be cylinder with a complex structure that I have explained earlier or it has to be complex torus with again with the complex structure I had explained earlier.

So, these are the only possibilities and you know, of course, in all these cases the fundamental group is Abelian. So, I will try to prove this in my next lecture. I will stop here.