An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1 -dimensional Tori and Elliptic Curves

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Lecture - 16
The Riemann Surface Structure on the Topological Covering of a Riemann Surface

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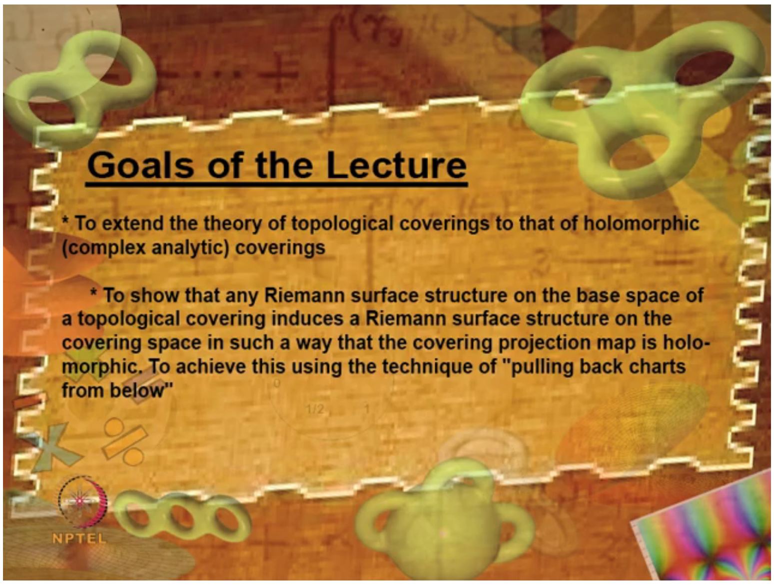

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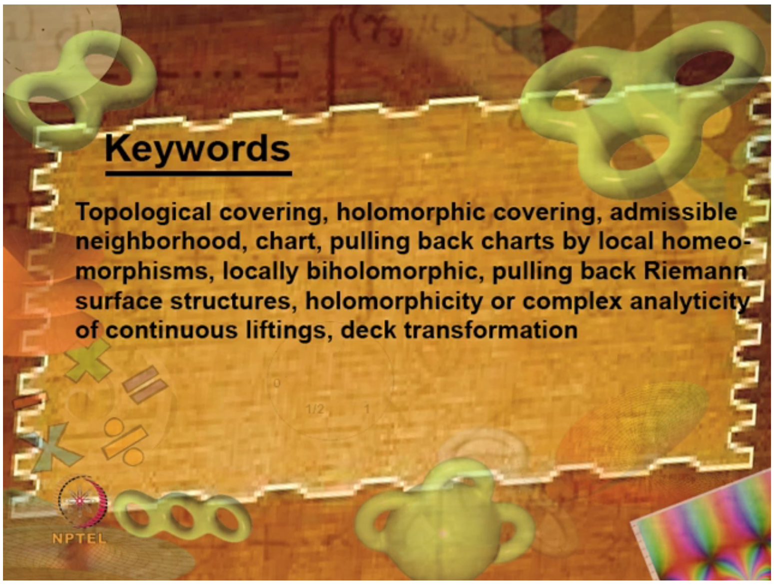

So we have seen in the last lecture how to construct the universal covering for a given topological space, and now we go back to looking at a Coverings of Riemann surfaces.

So, see so far whatever I have discussed about the coverings of a space they were all in the topological category. So, the maps involved were just continuous and therefore, and also the isomorphism that were involved were all homeomorphism they are isomorphism in the topological category, I just want to say that all these things will now go on in to the domain of the holomorphic category. So, what I am going to basically in this lecture is trying to explain the holomorphic theory of coverings the theory of holomorphic coverings right.

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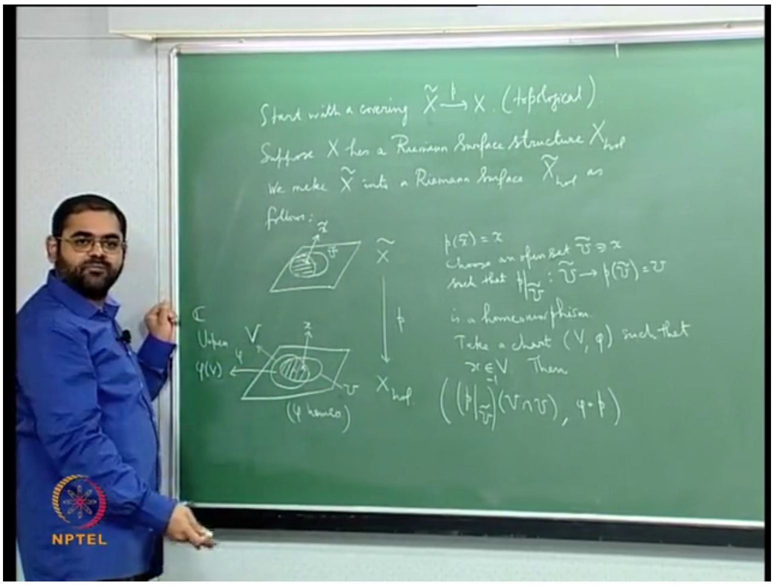

So, what I do is start with the covering with the covering X tilde to X. So, this is a so of course, when I say covering I mean topological covering topological covering right. Now I am going to look at the situation when this the base of the cover, has this structure of Riemann surfaces.

So, suppose X has a Riemann surface structure which I which let me call it as X sub hole. So, X is a topological space, but X sub hol is the topological space endowed with the Riemann surface structure, which means you know what it means it means every point of X is given coordinate chart it is given a chart that is a neighborhood which is homeomorphic to an open subset of the complex plain and all these charts a 2 such charts that intersects are compatible namely the transition functions are holomorphic and of course, when I take the Riemann surface corresponding to this I mean I take I include all possible compatible charts. So, I will take the maximal atlas. So, that is what I denote by X sub hol.

Now what I am going to explain is that this X tilde naturally gets a again a holomorphic structure and makes a this into Riemann surface and this map becomes holomorphic all this happens very naturally just because p is a local homeomorphism. So, now what I am going to do is I am going to trying to give charts on X tilde. So, the we make X tilde into a Riemann surface X sub hol X tilde sub hol as follows. So, what I am going to do is so will again let me draw a diagram.

So, you see I have a X tilde here and I have X this is p and here is my here is my X and well I take a point take a point of X tilde small X tilde, then you know pretty well that if I take the images at point below and suppose I call it as X. So, p of X tilde p of X tilde small x tilde is small x, then you know that because this is a covering in fact I do not even need this covering, but anyway it is a covering I can find a neighborhood an admissible neighborhood for X. So, if I call that admissible neighborhood as u then it is inverse image the inverse image of this admissible neighborhood under p, will give you several copies of this neighborhood above and exactly 1 of them will contain X.

So, what I will get is I will get I will get u tilde which is 1 1 of these copies. So, for this I need not even taken admissible neighborhood below infact even if I knew that this is not a covering, but if I knew that only there is a local homeomorphism I can always taken an open set above which is going to be mapped homeomorphically onto it is image below which is an open set by this map p. So, I can do I am saying that I can do this even p is local homeomorphism. So, in any case we can choose and open neighborhood open set u tilde containing X such that that p restricted to u tilde from u tilde to t of u tilde which is u is a homeomorphism I can do this; I can do this even if p was just a local homeomorphism at the point X tilde.

And now you see the space below X has the Riemann surface structure. So, let me write this as a X sub hol let me let me write this as a X sub hol just to tell you that that is a Riemann surface structure below so; that means, that at every point there is a chart there is a coordinate chart. So, what happens is that I can find. So, there is an open neighborhood of this point let me call that as b I have an open neighborhood of this point and I have a chart namely a function from this into phi which is which goes into phi of v, which is an open subset of the complex plane and phi is an homeomorphism.

So, this is the this is just the chart at the point X because the space below has a Riemann surface structure there is a chart given at each point I am taking a chart V comma phi which contains the point X alright. So, take a chart V comma phi such that small x belongs to V that small x is the image of X tilde.

Now, you see if I look at this intersection this, this intersection this, this intersection. So, let me fade it this point is x this shaded region is V intersect U. If I take this V intersect U and if I take it is inverse image under p recitative U tilde what I am going to get is am going to get again here I am going to get shaded here and of course, my point X tilde will be here and from u tilde to U sub it is a local homeom it is a homeomorphism p recita p recitative tilde.

So, recitative to this shaded portion till to this shaded portion will also be a homeomorphism alright and if you come back if you combine that with this phi, I will get a chart in this neighbor for the space above at the point X tilde. So then, you know how do I right that set theoretically I just I just take first of all the intersection of V and U which is a shade shaded portion here V intersection U, then what I do is I take it is inverse image under p recitative U tilde mind you I can write inverse because p recitative U tilde is a homeomorphism.

So, I can write the inverse. So, that gives you me this shaded region that is that is an open set containing x small x tilde and I have to and I and the map I take is p followed by phi. So, p followed by phi this of course, phi followed by when I say phi followed by phi that recitative to this shaded region of course, then this becomes a chart at X tilde because it is this is a homeomorphism followed by another homeomorphism. So, it is a homeomorphism into again into an open subset of where the complex main. So, it is a chart. So, basically what I am doing is I am pulling back charts from below I am just pulling back charts from below.

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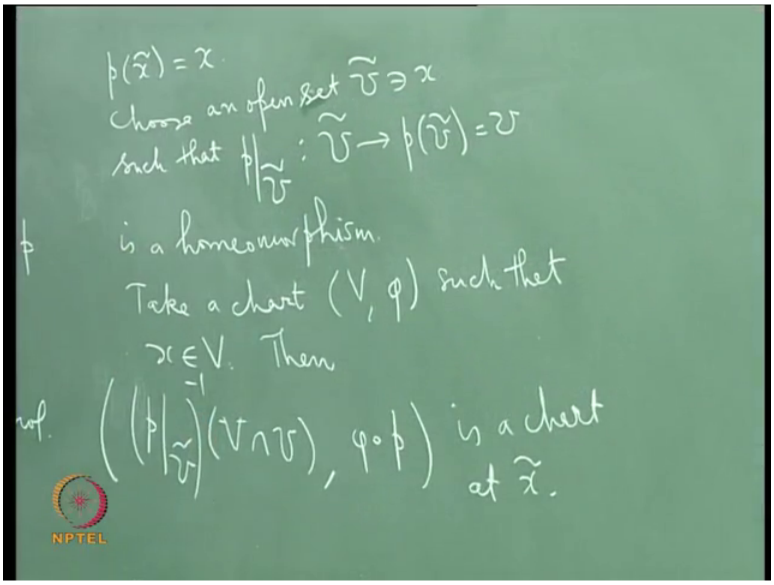

So, then this is a chart for at x tilde, Now, it is it is kind of pretty clear that all charts of this type they are going to be compatible and the and the reason is that the if you write it out the transition functions are going to be actually the transition functions for the charts below. And once we know that then you know that X tilde has Riemann surface structure which is what I am going to call as X tilde sub hol. So, let me try this down. So, well 2 such charts.

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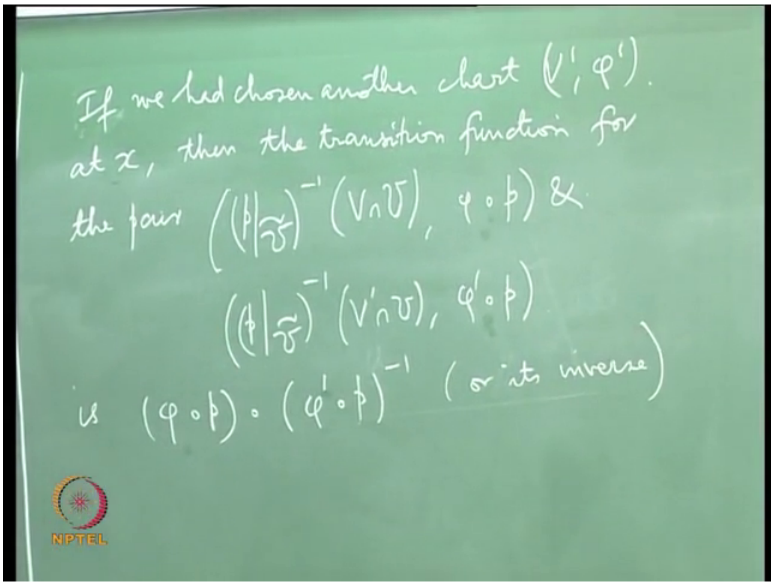

So, instead of doing that let me if we had chosen another chart V prime comma phi prime at x, instead of V comma phi suppose I chose another chart then; then the transition function for the pair the pair this chart namely. So, this chart, p recitative U tilde well inverse, V intersection U comma phi circle p and it is similar thing that you will get with by pulling back the chart V prime comma phi prime V prime comma phi is what is the transition function that transition function is just this followed by that inverse or that followed by this inverse.

So, if you give me pair of charts there are 2 transition functions they are going to be both you know homeomorphisms and the condition for compatibility is that these should be actually holomorphic that is your condition, when do we say 2 charts are compatible we say 2 charts are compatible whenever if they intersect that intersecting opens at is mapped by both of these map chart maps to different domains onto the complex plane, and then the map from this 1 of these domain into the other, which is given by taking inverse of the maps corresponding to 1 chart followed by the map corresponding to the other chart, you want that not only that will of course be a homeomorphism you want that to be holomorphic that is the condition for compatibility of charts.

So, what will happen is in this case the transition function will be phi circle p well if the inverse of the 1 followed by the by the other or of course, you know or it is inverse or it is inverse depending on which order you take it in alright, but what is this, this is just you know, if you write it down it is phi circle p circle p inverse circle phi prime inverse.

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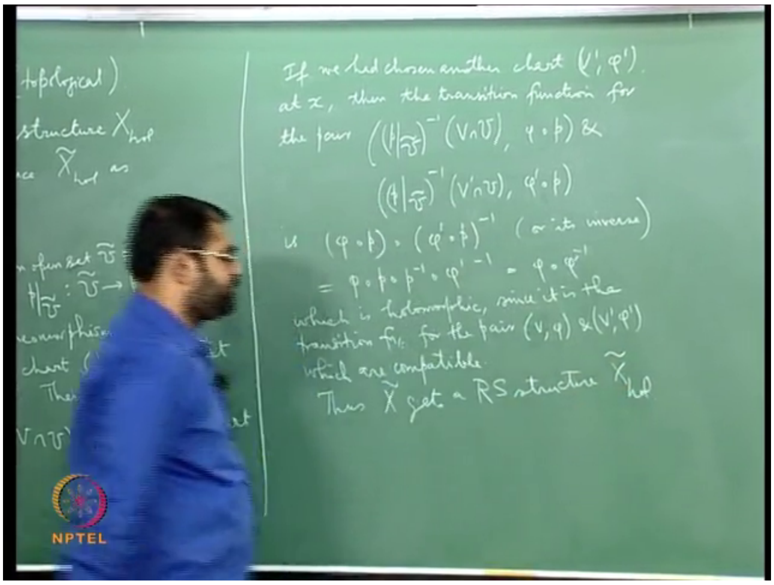

So, you will simply get phi circle phi prime inverse, but what is this? This is a transition function for these 2 charts per V comma phi and V prime comma phi and that is holomorphic because they are compatible because they define a Riemann surface structure below. Therefore, you see the way that you have given the charts automatically they will be compatible and you will get a holomorphic structure Riemann surface structure on x tilde, why it will still be a surface is because you see it is a local homeomorphism. So, dimension is not going to change locally they are isomorphic. So, this if this is surface that is also going to be a surface.

So, which is which is holomorphic since it is the transition function for the pair of charts the pair V comma phi and V prime comma phi prime which are compatible . So, the moral of the story is that all the charts we are going to get we are going to give you they are all going to be compatible. And therefore, x tilde will get Riemann surface structure which are I will call as a x x tilde sub hol thus X tilde gets a Riemann surface structure which I will call as X tilde sub hol.

Now, So, you see what is happened is that the Riemann surface structure below has been transported to there a to a Riemann surface structure above and that has been done by this that has been done by this map because it is a local homeomorphism on, I do not think I am even using that it is a covering alright the only thing I need is that it is a local homeomorphism. Now once you do this you see the beautiful thing that happens is once you do this. So, you can think of p as a local homeomorphism it is a map from x tilde whole to x hol and then the beautiful is that p becomes holomorphic, but you see if it is holomorphic and locally homeomorphic; that means, it is locally it is holomorphic and locally injective and open infact it is holomorphic and locally injective.

But you know injective holomorphic map is open an injective holomorphic map is open and it is by holomorphic. So, what will happen is that p will not only become holomorphic it will become locally by holomorphic. So, the local homeomorphic the p becoming p being a local homeomorphism that property will immediately give you the p is a local biholomorphism.

And therefore, what happens is that if you give me a admissible neighborhood below, then not then of course, the you take the inverse image of this admissible neighborhood you are it is going to break down into neighborhoods above open sets above such that p recitative that is not only going to be a local is not only going to be a homeomorphism, it will actually be a holomorphic isomorphism it will be a biholomorphic map. So, what will happen is that this whole covering will become completely a covering in the holomorphic sense.

So, just because the fellow here the base space here is a Riemann surface. So, that is what I want to I want to stress upon.

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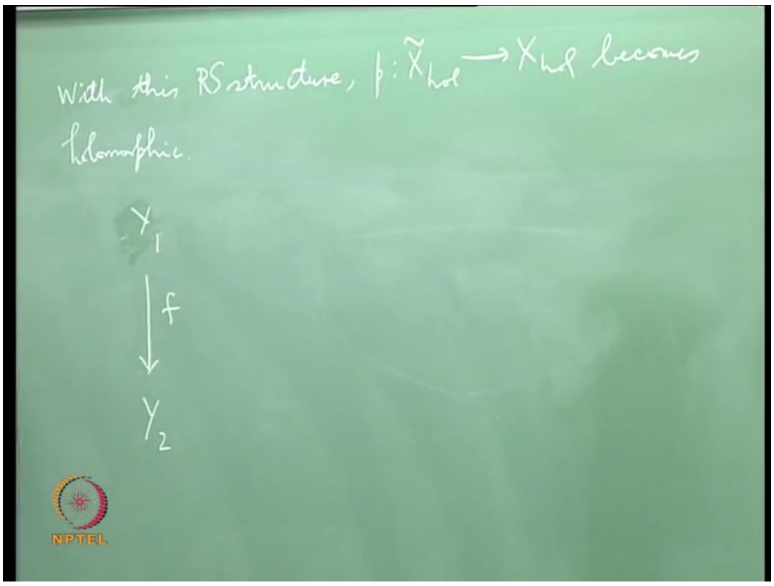

With this Riemann surface structure p from X tilde hol to X hol becomes holomorphic, why does it become holomorphic what how do you check that you see normally you know if you give me; let me use Y 1 and Y 2 suppose they are Riemann surfaces and f is a map from Y 1 to Y 2, when do you say that f is holomorphic the method is that what you do is that you check this holomorphic at each point. So, first of all f has to be continuous.

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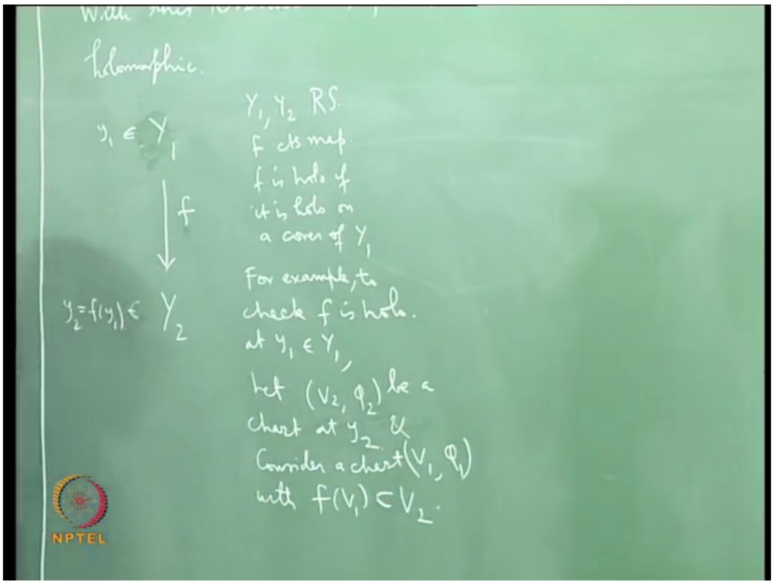

So, well Y 1 comma Y 2 Riemann surfaces F continuous map f is holomorphic if it is holomorphic on a cover of Y 1 to check you know to check properties like continuity differentiability etcetera nice properties it is enough to check them on a cover, if you want to check them on a whole space alright. So, if I want to check map is holomorphic I can check it check then it is holomorphic with respect to an open covering and what covering do I take I take the covering of charts I take a chart here I take a chart there and I check and how do I check well you can even check it point wise. So, you know if Y 1 is a point of Y Y 1 and if it goes to a point Y 2 which is f of Y 1.

For example to check f is holomorphic at y 1 what you do well you do the following thing you take you take a charge centered at Y 2. Let us say V 2 comma phi 2 be a chart centered at a chart let me say when I say centered I mean containing a chart at chart at y 2 for the Riemann surface below alright and consider a chart a chart V 1 comma phi 1 with you know f of V 1 going into V 2 . So, I can I can do this you see V 2 is a is an open set it contains y 2 by continuity I must be able to find open set V 1 of y 1 which goes into y 2 that is just because of continuity and then I can it can be it will have a there will be a chart if you may be I may have to shrink Y 1 if necessary. So, that I get a chart. So, once you have once you fix this how do you check that the map f is holomorphic you saw what you do is you do the following thing.

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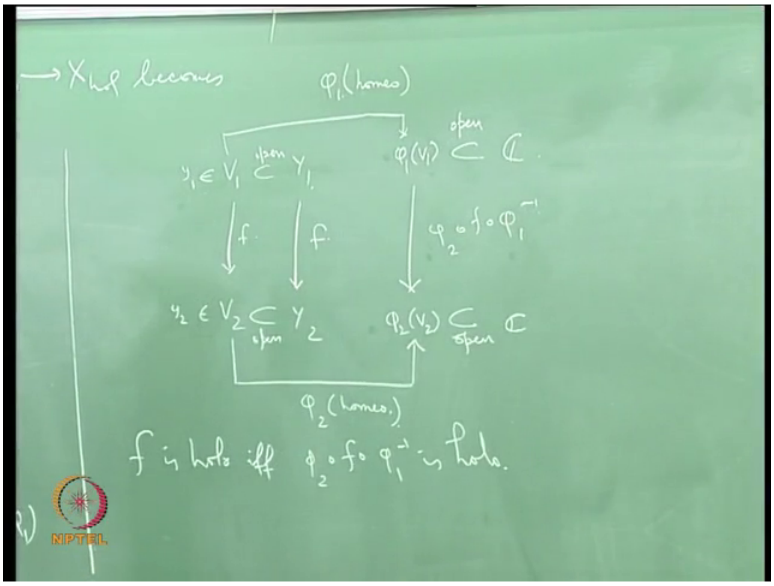

So, you have Y 1 here which is sitting inside V 1 which is sub which is an open sub of Y 1 and f is going to be go from Y 1 to Y 2 it is going to take V 1 to V 2 this also open and this contains y 2.

Now, you see from V 1 there is chart V 1. So, what is going to happen is from V 1 to. So, you have phi 1 which is a homeomorphism on to phi phi 1 of v 1 which is an open subset of the complex plane and well from V 2 to you know phi 2 of V 2 you have this phi 2 it is a homeomorphism onto phi 2 of V 2, which is an open subset of the complex plane and now what you do is you go like this; namely you apply phi 1 inverse then apply f and then apply phi 2. This will give me a function from an open subset of the complex plane to another open subset of the complex plane and all I have to do is a I have to check that this function is holomorphic if I want to check a f recitative even if holomorphic I have to check that this function is holomorphic that is the definition.

And if I want to check F is holomorphic at y 1 I need to check that this function is holomorphic at the point that y 1 goes to here. So, let me write that f holomorphic if and only if phi 2 circle f circle phi 1 inverse is holomorphic this is the definition this is the definition of a map being holomorphic you check the holomorphicity in terms of local coordinates local charts. Now play this game with this situation and you will see it is very trivial.

Let us play this game with let us play this game with that. So, what is happening is you see I have I have this Shaded neighborhood.

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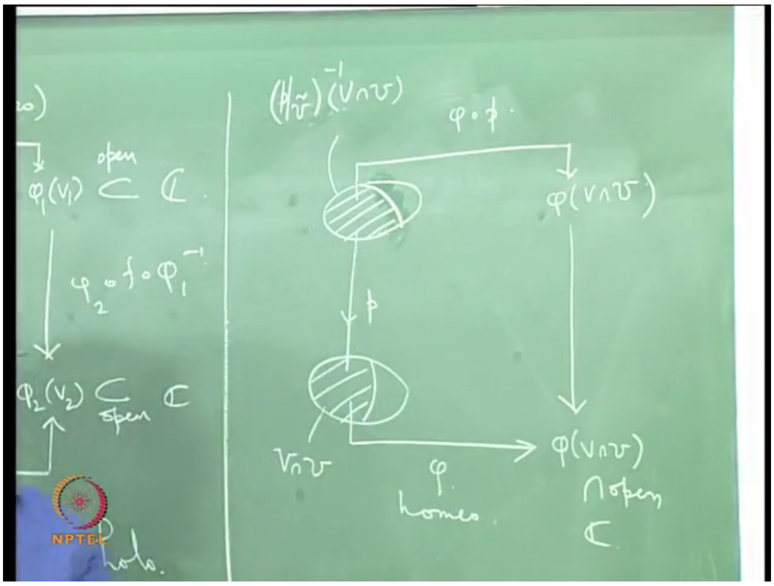

So, let me let me redraw the diagram there. So, you see I have the shaded neighborhood above and this shaded neighborhood is just. So, let me you know. So, this shaded neighborhood is U I guess u tilde p restricted p tilde inverse of v intersection u and then that is this map p from that into this space below into the open set below and this open set below is just v intersection u, and from here I and from here I have the I have phi which is the which is the chart below v comma phi is the chart below and it is going to take v intersection u to phi of v intersection u which is which is of course, an open subset of the complex plane and is a homeomorphism.

And on the other hand what is the chart here chart here is nothing, but I mean the chart a is actually this followed by this. So, the chart is here is just if you look at this definition it is phi followed by phi. So, chart above is phi phi p followed by phi and what it is image it is image is the same thing you apply p first to this to this shaded piece it will go this shaded piece and then apply phi you will get this. So, what I will get here is again phi of phi intersection u and if I write out this map as I have written write written it out here you know this map will just be the identity map this map will just be the identity map, because it will be if you go by this definition it is this map is going to be what is this map it is going to be phi p inverse followed by p and then followed by phi by this definition and what is this? This is exactly identity. So, this is just identity and the identity map is of course, holomorphic.

So, trivially the projection the map p becomes holomorphic. So, it is just I mean without know extra effort it is just come just like that. So, you moral of the story is therefore, p is holomorphic map. So, it is it is very clear.

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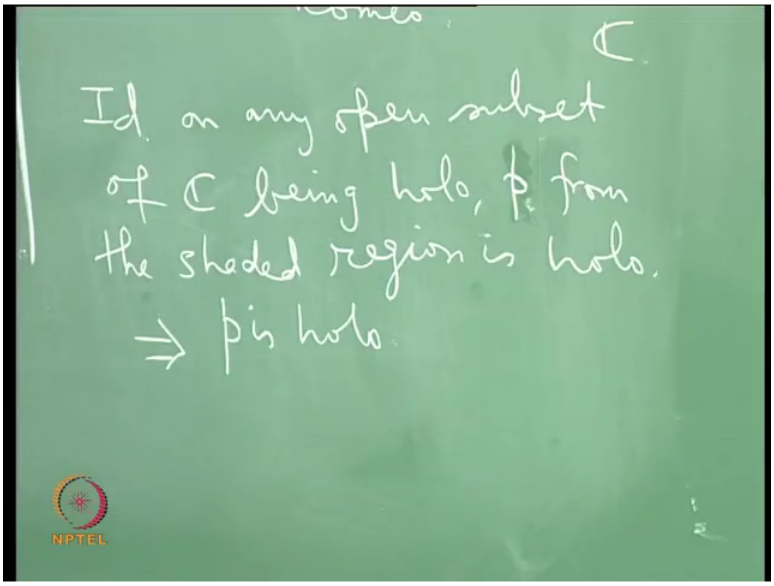

So, identity map on any open subset of C being holomorphic p restricted to this set let me call this is as or let me write it in words p from the shaded region is holomorphic alright so, but I can cover the space above by such shaded regions. So, we will be holomorphic for holomorphic everywhere to check something is holomorphic I have to check only on a cover.

So, p restricted to the shaded region is holomorphic and that will imply that p is holomorphic infact what will tell you is it is p region is holomorphic and that will imply that p is holomorphic in fact what will tell you is it is p is locally holomorphic if a global map is locally holomorphic then it is holomorphic, because that is I mean that is 1 of the good properties of I mean in fact that is how good properties will behave to say some is to say to check something is continuous you check it is locally continuous. To check that a map is continuous you check it is locally continuous and the same thing holds for something being holomorphic holomorphicity is a good property you can check it on a open cover.

So now, what is the up short up short of this is that, now since an injective holomorphic map is all is a is a biholomorphic map what this will tell you is that this is actually locally by holomorphic. And therefore, this is actually a covering in the holomorphic sense which means you take any admissible neighborhood below, the inverse image will be break down into disjoint unit of open sets not only recitative which p is a homeomorphism, it will actually be a holomorphic isomorphism. Namely; it will become a recitative to each of this pieces above p will actually be a bi holomorphic. So, it is completely in the holomorphic category. So, get a holomorphic covering alright because this becomes the holomorphic covering.

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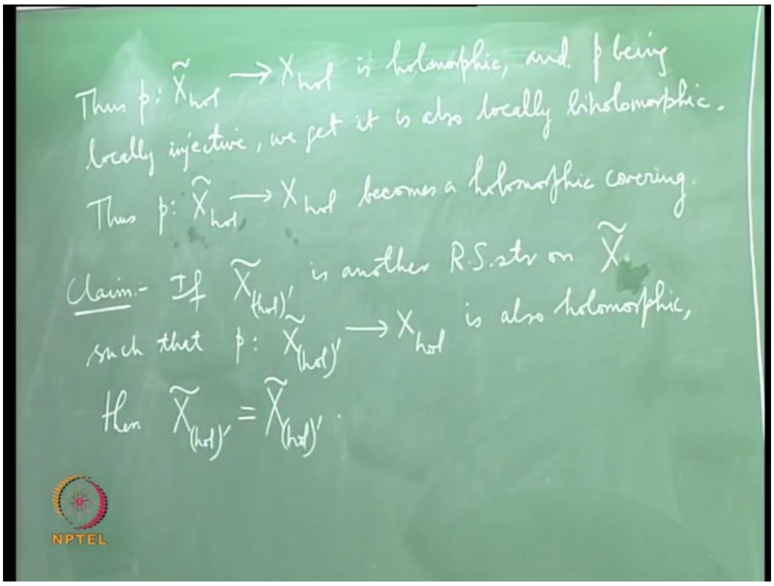

So, let me write that down thus P from X hol sub tilde to X of hol is holomorphic and p b locally injective, because it is locally homeomorphism p is locally injective we get it is also locally biholomorphic which means locally a holomorphic isomorphism p was in the topological case p was only locally a topological isomorphism topological isomorphism namely a homeomorphism. Now everything has transferred to the holomorphic category. So, what has happened is that p is locally even a holomorphic isomorphism which is what we call as a biholomorphic map. So, also locally biholomorphic thus p from X hol tilde to X hol becomes a holomorphic covering.

Now, what I want to say next is that I want to say that the. So, I want to say the following I want to say that you see I want to say that any other Riemann surface structure on X tilde, that makes p into holomorphic map has to be none other than this one. So, what I am saying is that you cannot get another holomorphic structure on X tilde that converts it into Riemann surface in another way, which is not the same as this.

So, there it is rigidity I mean the under this p to be holomorphic then there is only 1 Riemann surface structure on the covering it is unique that is what I am going to prove. So, claim if X hol let me put prime is another Riemann surface structure on X tilde sub X tilde such that p from that structure X tilde sub hol prime to X hol is also a holomorphic, then I will put then the then X tilde hol prime has to be equal to X hol prime tilde.

In other words there is only 1 Riemann surface structure on X tilde which makes p holomorphic there is only 1 there is a unique Riemann surface structure on X tilde that makes (Refer Time: 33:38).

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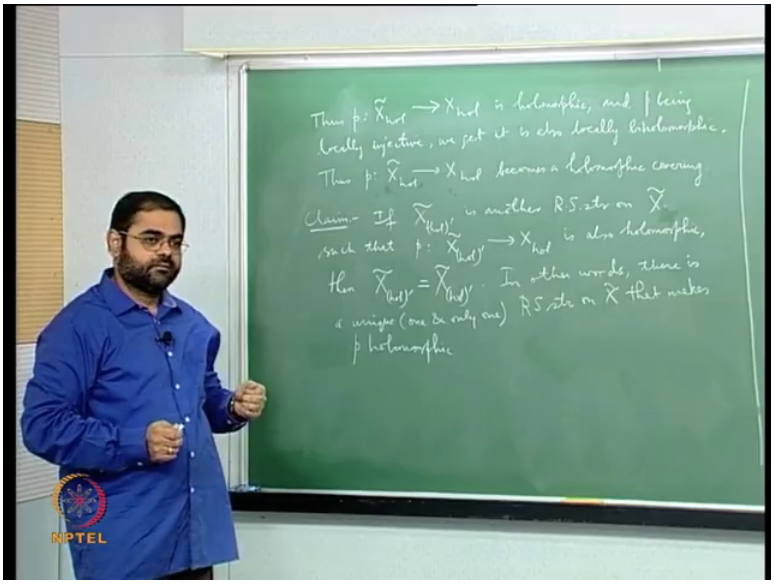

In other words there is a unique 1 and only 1 Riemann surface structure on X tilde that makes p holomorphic. So, this is the uniqueness of the Riemann surface structure that you get above. So, you see when you take a covering topological covering if the base space has a Riemann surface structure that dictates the Riemann surface structure on the cover. So, that is what happens. So, how does we how does 1 see this. So, once is this is follows. So, to see this I just again let me write down simple exercise.

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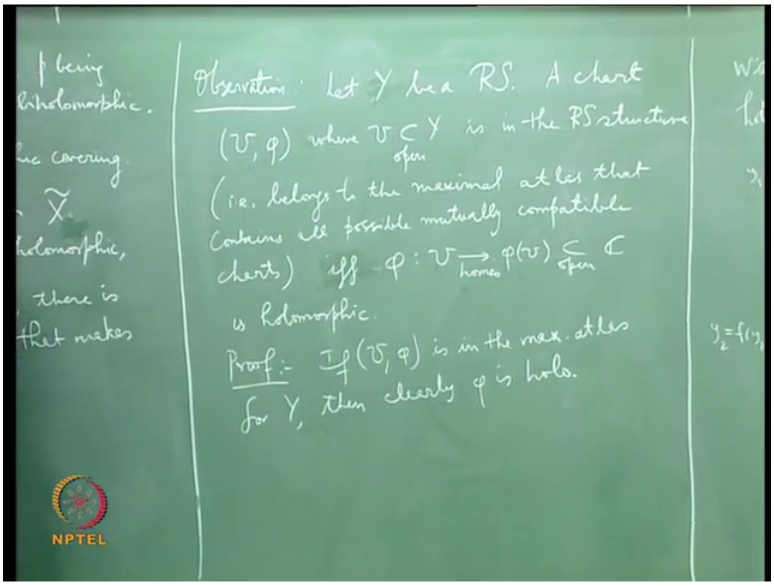

So, we will make an observation let Y be a Riemann surface that y be a Riemann surface so; that means, Y is given with a maximum atlas of consisting of all possible compatible charts collection of compatible charts. A chart u comma phi where u in Y is an open set is in the Riemann surface structure, that is belongs to the maximal atlas that contains all possible mutually compatible charts.

So, I have a Riemann surface and I have chart the question when this chart in that Riemann surface in the in the maximum in that Riemann surface means in the maxima it will defined by the Riemann surface and I am saying the condition is just phi has to be holomorphic then . So, I have not completed by status a chart u comma phi where v. So, where y is open is in the Riemann surface structure, if and only if phi from u to phi of u which is an open subset of C this is a homeomorphism is holomorphic.

So, this is my remark this is my remark. So, please try to understand what I am saying y is Riemann surface alright u is an open subset of the Riemann surface and phi is just a homeomorphism of u into an open subset of the complex plane, when will this also be compatible with every chart that defines the Riemann surface structure on y that is the question.

I am saying that will happen if and only if this phi itself if holomorphic. So, if you remember I told you that 1 way is I told you why 1 way is very easy. So, what is the proof for this the proof for this is if u comma phi is in the maximal atlas for Y then clearly phi is holomorphic, because if you remember I told you long back in 1 of the earlier lecture that you see how do we define the Riemann surface we first define Riemann surface by giving a cover by chart and the charts are only the homeomorphism of open subsets of the surface into the complex plane the only condition you put is that they transition functions are going to be holomorphic.

By the way I told you once you have defined the Riemann surface structure then the coordinate map given by each chart that itself becomes holomorphic. So, if u comma phi is in a chart is in this maximal atlas then clearly phi is going to be holomorphic I have to prove the other way round. So, I hope I already done this exercise.

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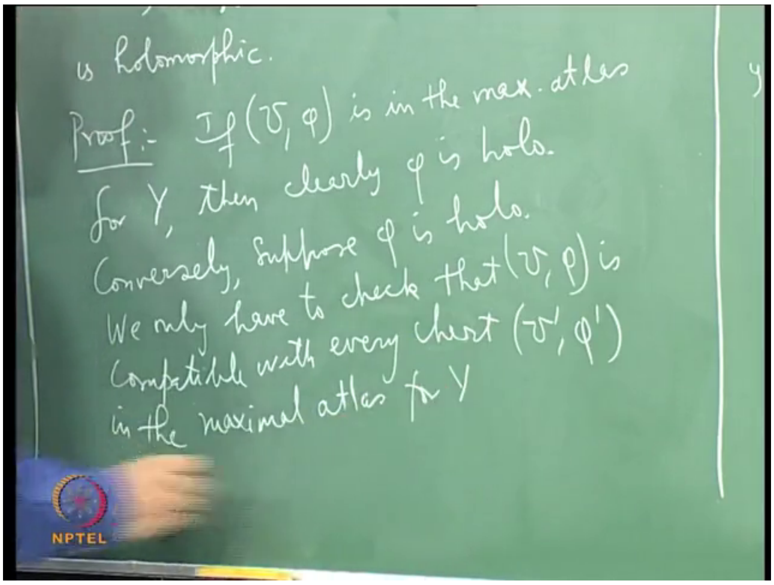

Conversely suppose phi is holomorphic conversely suppose phi is holomorphic I will have to check only that u comma phi is compatible with every other chart that intersects it. So, we only have to check to check that u comma phi is compatible with every other chart every chart u prime comma phi prime in the atlas in the maximal atlas for Y. In fact you need not check with maximal atlas you can check with a an atlas which consists of covering. So, why is that? So, you can easily check it. So, let me write that down.

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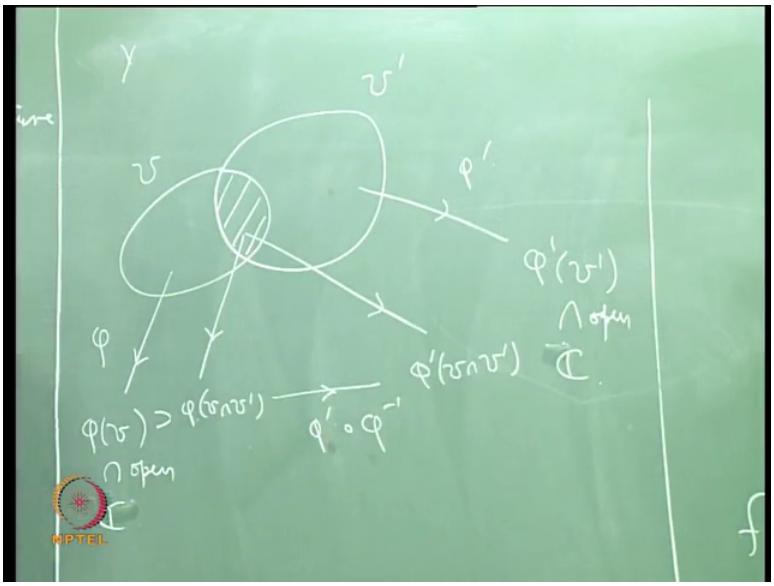

So, you have the situation is like this you are on Y and there is this u and there is a phi which is a homeomorphism into phi of u which is a open which is an open subset of the complex plane and there is your given another u prime which is also an open subset and a map phi prime, which is homeomorphism onto phi prime of u prime, which is again an open subset open subset of the complex plane and how do I what is the compatibility of u comma phi with u prime with phi prime I will have to take this intersection.

And I will have to write out the transition function and say that that is holomorphic that is my condition. So, what is the transition function? So, the transition function is going to be like this. So, this is going to go to phi of u intersection u prime and here it is going to go to phi pi prime of p intersection u prime and my transition function is going to go like this what is the transition function it is just apply phi inverse and then apply phi prime this is my transition function and where are these 2 charts are compatible, when this holomorphic when this holomorphic. So, that is what I have to check; I have to check this holomorphic, but what is given to me what is given to me is that phi is holomorphic.

So, phi is holomorphic therefore, phi inverse is also holomorphic phi is holomorphic in phi is injective. So, it is biholomorphic. So, the inverse is also holomorphic. So, this is holomorphic and phi prime is holomorphic because u prime comma phi prime is already in the atlas anything in the you take any coordinate in the atlas the coordinate map for the chart is holomorphic. So, phi prime is also holomorphic. Therefore, this composition is holomorphic. So, I will check the compatibility.

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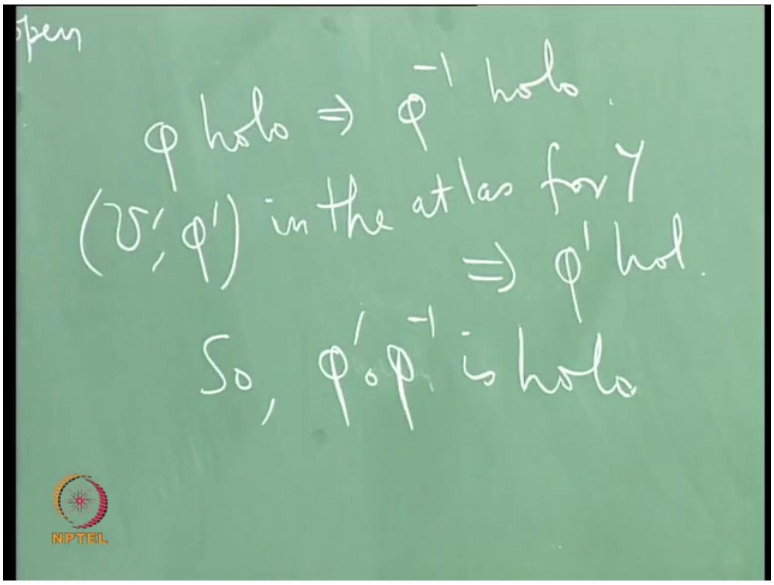

So, phi holomorphic implies phi inverse holomorphic U prime comma phi prime in the atlas the atlas for Y implies phi p prime is holomorphic. So, phi prime circle phi inverse is holomorphic. So, this completes the proof of this claim.

So, let me again some of this observation to check that a chart belongs to a Riemann surface structure all you have to check is that the coordinate map of the chart is holomorphic nothing else. Now keep that in mind and I will tell you why there is only 1 Riemann surface structure on X on X tilde that makes p holomorphic. So, let me rub this I hope I would not need it again, but so now, let us apply this observation to let us apply this observation to this map. So, what is given is a so you see coming back to our situation.

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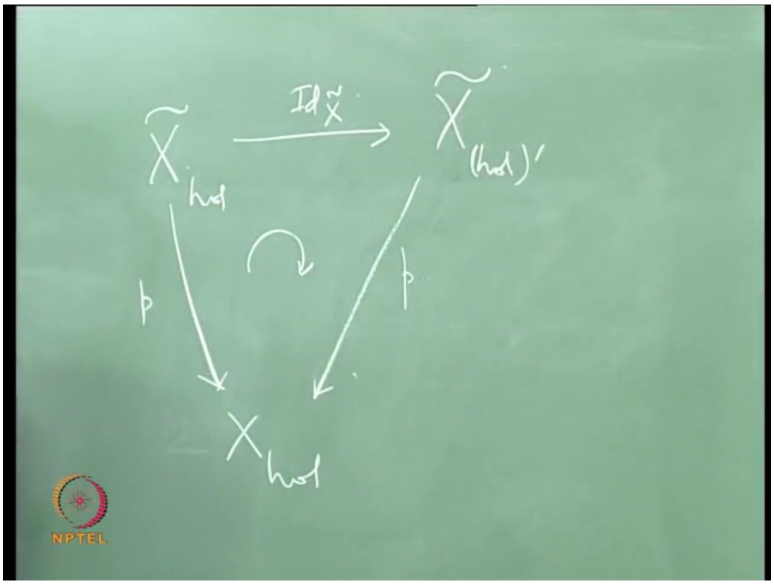

So, I have seen I have X tilde hol to X hol this is p this is this is already holomorphic alright and what is given is that there is a X hol prime it is another Riemann surface structure on X tilde and then this is also holomorphic.

Now, what I am going to do I am just going to take the identity map take the identity map this diagram commutes because this is just identity followed by p is just p set theoretically just take the identity map of X tilde the identity map on X tilde is of course, it is of course, a homeomorphism no doubt about that the point I want to make is that this identity map on X tilde is locally biholomorphic that is because you see this is locally biholomorphic and this is also locally biholomorphic.

So, you know I can I can go from a neighborhood here to an isomorphic holomorphic isomorphic neighborhood here and then I can go to go back to the same neighborhood under this map. So, what this will tell you p both these p s being a locally biholomorphic will tell you that the identity map on X tilde is locally biholomorphic; that means, it is; that means, identity map itself is holomorphic I told you if the map if a map is locally holomorphic then it is holomorphic.

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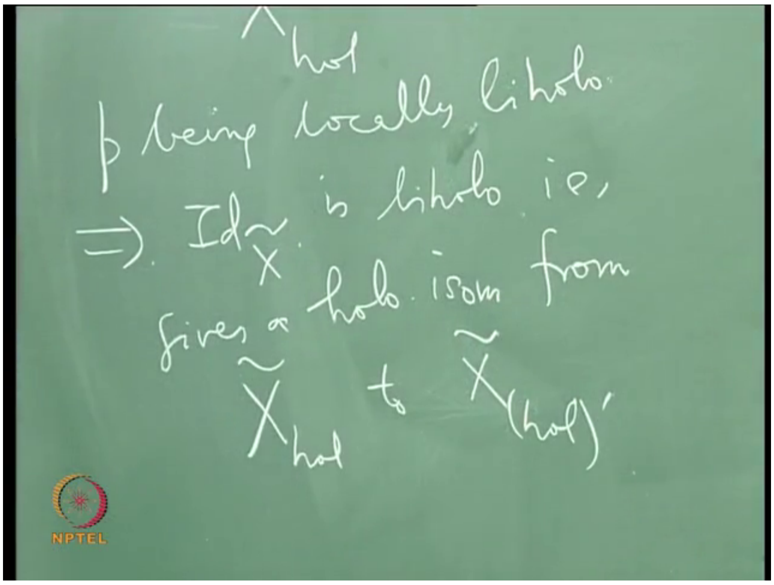

So, p being locally biholomorphic implies that the identity map on X tilde is biholomoprhic. So, is biholomorphic (Refer Time: 46:31) I mean. In fact, it is a it gives you holomorphic isomorphism for this to that that is gives a holomorphic isomorphism from X tilde hol to X tilde hol prime.

Now, this should tell you that these 2 are the same if you have any doubt you just have to you just have to apply this. So, what I will do is let me take any chart there and tell you it is already here.

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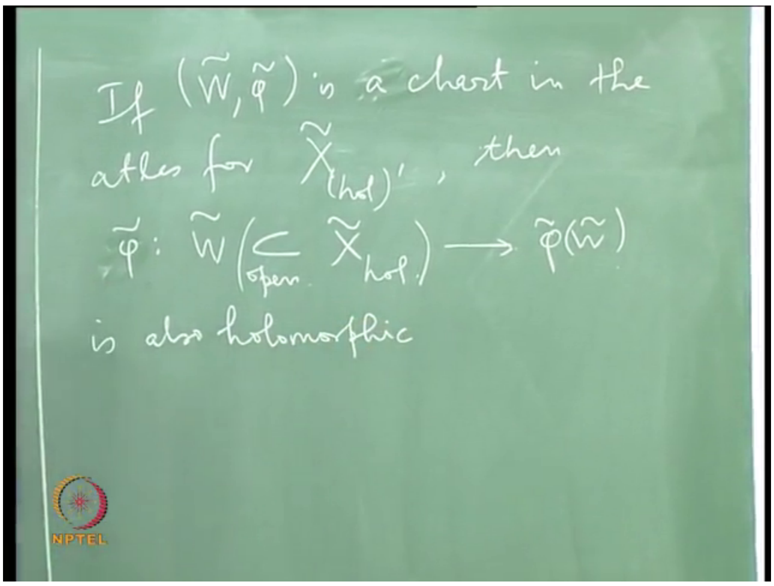

See if let me say w tilde phi tilde is a chart is a chart in the atlas or this guy X tilde hol prime, then you see W then, then phi tilde from considered as a map from W tilde considered as an open subset of X tilde hol to phi tilde of W tilde is also holomorphic, why because you see what is happening is you have X tilde hol.

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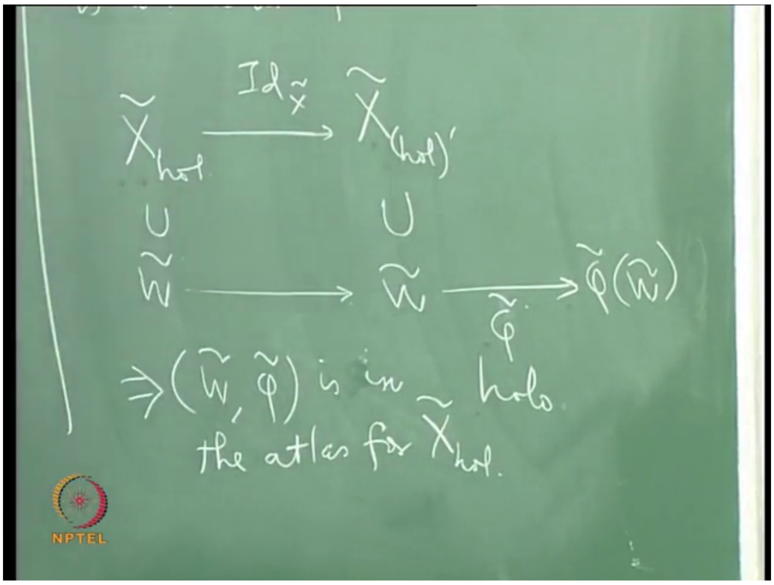

And I have my w tilde then I have this identity map on X tilde, which is holomorphic and that it carry w tilde back to itself as an open subset of X tilde hol prime and of course, since phi phi tilde is a coordinate map for a chart which is in X tilde hol prime, this map this is phi tilde into phi tilde of W tilde this is holomorphic this map is holomorphic, that is because phi tilde W tilde comma phi tilde is in the is a chart for X tilde hol prime. So, this map is holomorphic this map is just the identity map and we have checked that is already holomorphic answer. Therefore, the composition the composition is just p tilde considered as a map from W tilde as an open sub set of X tilde hol that is also holomorphic. And therefore, by this observation you get the this chart is also an X tilde hol; so you see.

So, this implies that W tilde comma phi tilde is in the atlas the atlas for X tilde sub hol. So, the moral of the story is that these 2 are actually equal they have the same atlas maximal atlas. So, therefore, the covering space of a Riemann surface inherits a unique Riemann surface structure such that the covering projection is holomorphic map and the whole projection becomes the holomorphic covering. So, that is the whole point and mind you all this started with just taking a topological covering and making the assumption that the base of the covering has a Riemann surface structure alright. So, that is what I wanted to emphasize.

So I just want to also make a couple of remarks to tell you that everything is really happening in the holomorphic category. So, if you remember this is connected to in the when we are looking at the topological theory of coverings, there was this unique lifting property unique path lifting property and we also had this identification as of the fundamental group of the base space with sub group of homeomorphism of the universal covering. Now I just want to say that all this will happen in the holomorphic category. So, let me explain how this happens.

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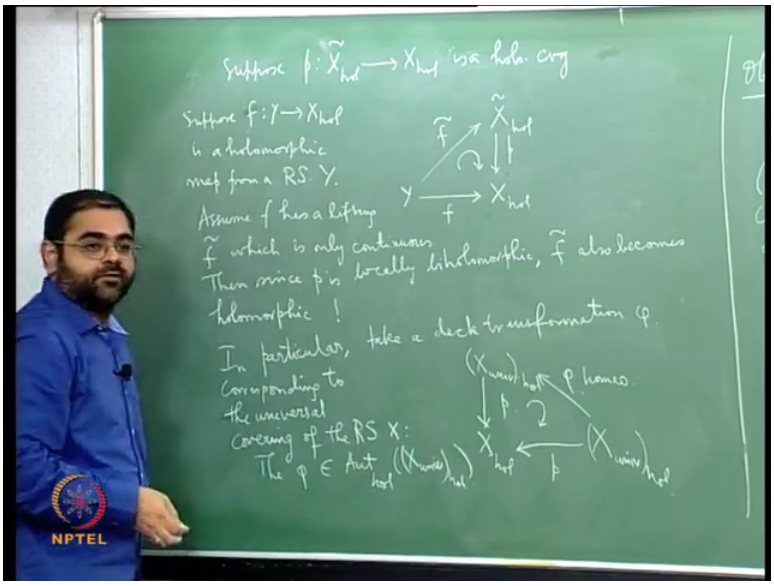

So, suppose I start with a covering map which is a holomorphic covering. So, suppose p from X tilde sub hol to X sub hol is a holomorphic covering. So, situation is like this. So, I have X tilde I have p I have X this is holomorphic this is also holomorphic; that means, these are Riemann surfaces and of course, p is the covering position.

Now, if Y is another Riemann surface and suppose from Y to X hol I have a map f which is holomorphic map. So, suppose f suppose Y from y 2 X hol is a holomorphic map from a Riemann surface Y. Now assume that there is a lifting assume that there is a lifting from f tilde namely this diagram commutes, but assume that f tilde is only a continuous lifting, assume f has a lifting, f tilde which is only continuous.

So, you look at what is happening I have a holomorphic map of Riemann surfaces and it is lift able to continuous map to the covering space then what happens is f tilde automatically becomes holomorphic. The reason is because f tilde can be locally it is f followed by an inverse for p and p is locally biholomophic. So, locally f tilde is holomorphic that is just because locally I can invert p and the inverse of p is also holomorphic.

So, the moral the moral of the story is if you get a continuous lifting for a holomorphic map it is automatically holomorphic it also comes into the holomorphic category that is a point. So, then since p is holomorphic is locally biholopmorphic f tilde also becomes holomorphic holomorphic. So, you know if you want to lift a holomorphic map it is enough to lift it only to a continuous map the lift will automatically become holomorphic you get that for free that just because p is locally biholomorphic that is one remark.

Now, what is the advantage of this remark the advantage of this remark is that if you take the universal cover, then the deck transformation group which was identified as a sub group of homeomorphic automorphism of the universal cover will now be identifiable as a sub group of holomorphic automorphisms of the universal cover given the natural holomorphic structure the unique structure that comes from below right. So, in particular you see take a deck transformation. So, you know take a deck deck transformation phi. So, I have X I have X sub u new and this is the universal cover alright take a deck transformation phi corresponding to corresponding to the universal covering of the Riemann surface X.

So, what it is it? So, X is a Riemann surface. So, I will write as X sub hol alright X sub u new is universal cover that also gets unique Riemann surface structure. So, that this becomes holomorphic covering and now you see what is the deck transformation see it is actually a life of p which is a homeomorphic.

So, it is again a map from X to X of univ which has a holomorphic structure and it is a map like this p such that this diagram commutes and phis is homeomorphism what is the deck transformation deck deck transformation is just a homeomorphism of the universal covering, which respects the projection. So, it is a lift of p, but it is ho it is a homeo morphisms. So, it is continuous and I told that since everything involved here is holomorphic this phi automatically becomes holomorphic and holomorphic homeomorphism is a biholomorphic map therefore, what happens is p is not only an automorphism of the topological space underlying the universal cover, it is also a holomorphic automorphism of the Riemann unique Riemann surface structure that you get on this universal covering space that is a whole point.

So, then phi is an automorphism a holomorphic automorphism of X sub univ hol. So, the moral of the story is if you take the universal covering of a Riemann surface alright then the deck transformation group which is the fundamental group, which is cannot which is identifiable with the identifiable with the group of the topological space below actually becomes the subgroup the holomorphic automorphisms of the universal covering with the universal covering we given the natural 1 and only holomorphic structure that makes p the covering map into the holomorphic map.

So, I mean these two remarks should tell you that we are completely in the holomorphic category. So, that that is why I am saying that this is the complete theory of holomorphic coverings. So, that is what I wanted to emphasize here.

So, I will stop here.