An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1 -dimensional Tori and Elliptic Curves

Dr. Thiruvalloor Eesanaipaadi Venkata Balaji
Department of Mathematics
Indian Institute of Technology, Madras

Lecture - 15
Part B
Completion of the Construction of the Universal Covering: The Fundamental Group of the base As the Deck Transformation Group

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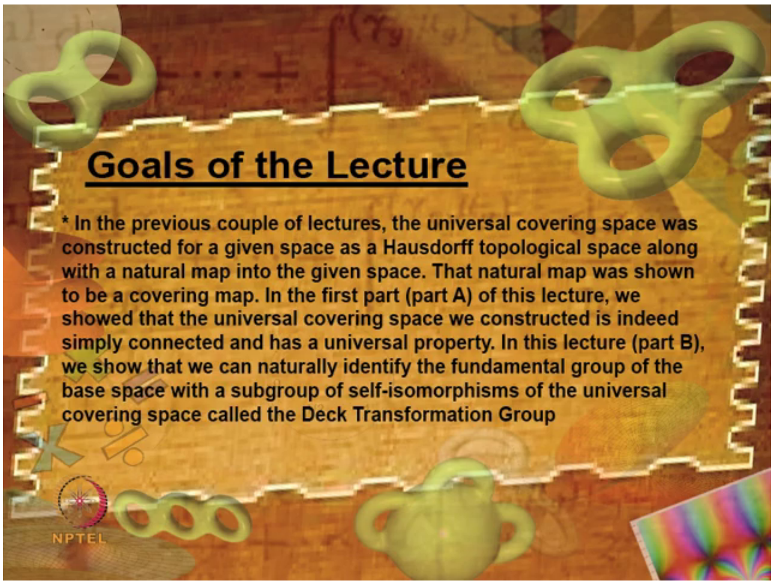

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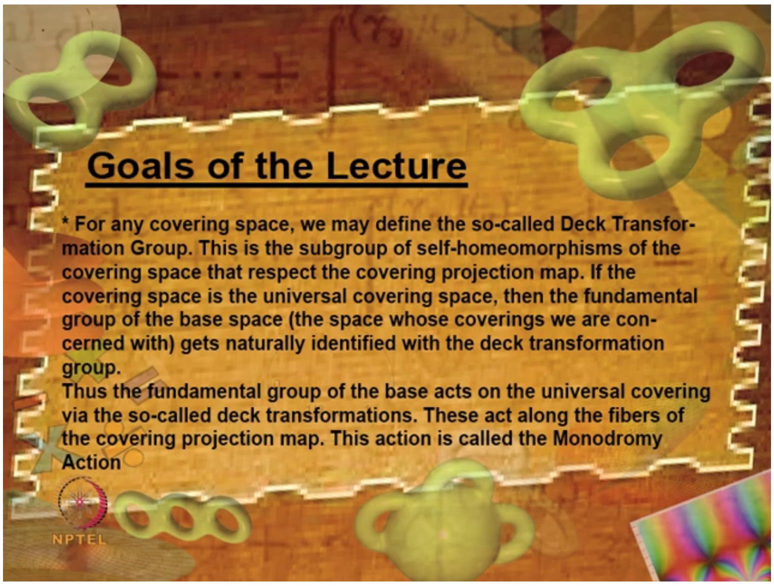

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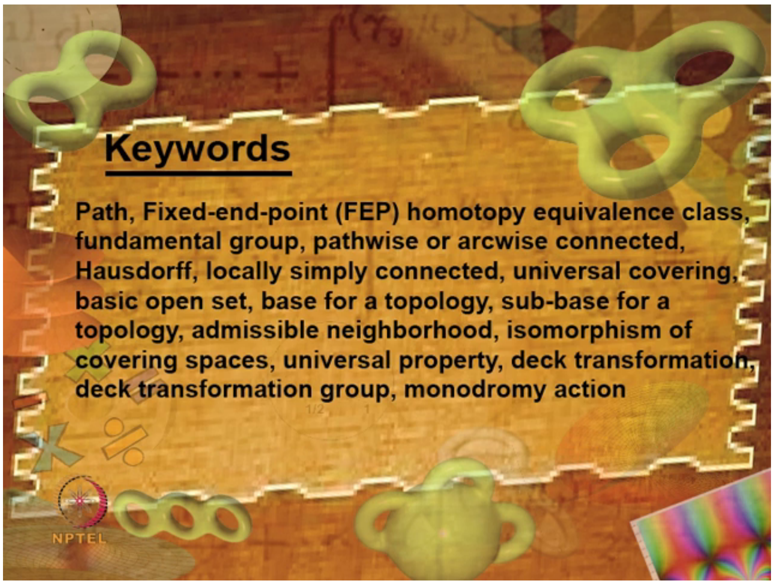

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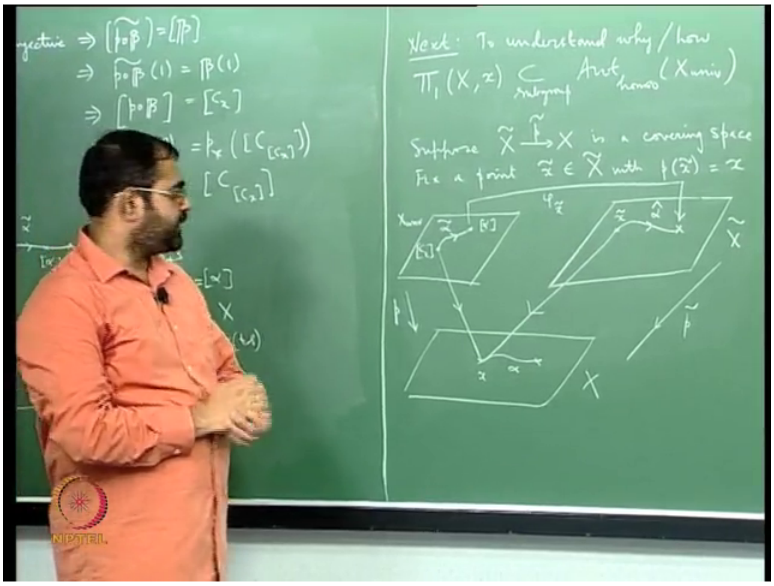

So, next to understand why or how the fundamental group of X comma x is a subgroup of the automorphism homeomorphisms of the universal covering space. So, pi is the fundamental group of the base space pi can it be identified naturally as a subgroup of automorphisms of the universal covering automorphisms which are of course, homeomorphisms self homeomorphisms.

So, for this one has to go through a small construction. So, sub; so we will do the following thing suppose X tilde to x p is a covering space you just take any other covering space of x fix the point. So, fix a point fix a point X tilde you know you capital X tilde with p of X tilde.

Let me call this as; let me call this is p tilde because I have used p already p is supposed to be our covering map from the universal covering to x. So, let X tilde from p tilde to x the covering space and you fix a point in a small x tilde and capital X tilde which lies over x. So, the situation is like this. So, here is x sub univ I fixed in x sub univ of course, I am going to take the natural point maybe the constant part is x which lies above which lies above x small x in capital X and then I am going to take another. So, there is another covering space X tilde given to me along with the covering map p tilde. So, this map is p and I have a point here namely small x tilde which is also lying over x under the map p tilde right.

Now, what I am going to do is that I claim that this much of data is enough to define a map from x from this universal covering to that covering such that this map itself becomes a covering. So, I am going to do the following thing; so going to do the following thing take any point of x sub univ say alpha.

Now, well this alpha well is going to correct is going to correspond to me to a path here alpha of starting at x and I have; so, you know and of course, for this alpha that is this there is this a very natural alpha tilde that we constructed which is given by the shrinking homotopic that is the path that lifts alpha to the universal covering right; what I am going to do is now since I have a path here and since this is a covering there is a lift there is a lift of this there is unique lift of this to a path starting at X tilde ok.

So, what I am going to get is I am going to get I am going to get a path here let me call this as alpha hat. So, this alpha hat is unique lift of alpha to X tilde starting at x and it has an end point I am going to define a map which will send this alpha to this point. So, what I am going to do is this is my map. So, this is my map phi and this map depends on this map depends on this X tilde that I have fixed there. So, let me write that down.

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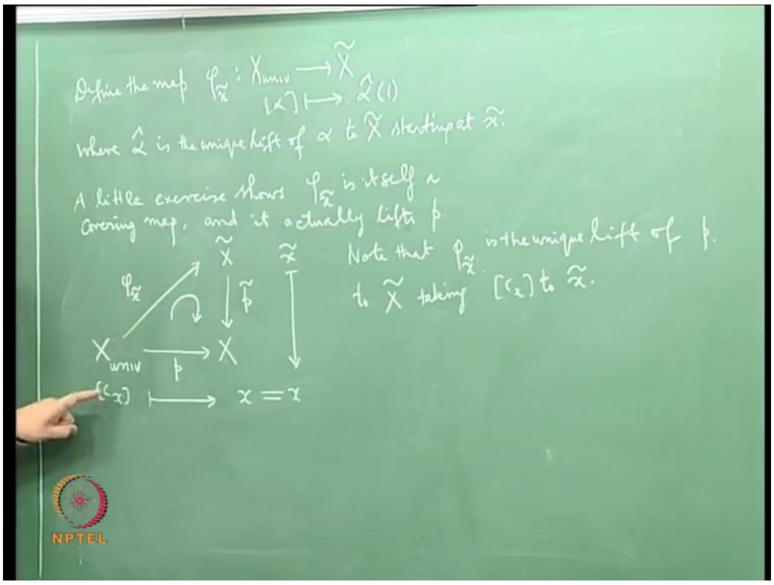

So, define the map phi sub X tilde from x sub univ to X tilde by sending alpha to alpha hat of one where alpha hat is the unique lift of alpha to X tilde starting at small x tilde. So, this is my map right.

Now, the point is that this map is well defined this map is well defined because if I change alpha by something that is homotopic to alpha. So, then; that means, I will change if alpha is also homotopic to beta, then I will have they will; then I will have a beta here which is fixed end point homotopic to alpha and then you know beta will also lift here, but then the covering homotopic on a theorem says that you know fixed end point homotopies can be lifted to fixed end point homotopies. So, the lift of beta hat of beta will also have the same endpoint as the lift of alpha hat. So, this map is well defined even if you change alpha up to homotopic fixed endpoint homotopic. So, this is well defined map.

Now, it is somewhat of an exercise basically using the fact that you know these are local homeomorphisms and they are covering maps a little exercise will tell you that this itself is a covering map. So, let me write; let me write that; let little work exercise shows phi sub X tilde is itself a covering map. So, this is some work that needs to be done alright you can verify that for yourselves alright and what I would like to say is that in some sense this whole thing is gives what is called as universal property of the universal cover.

So, let me say some something more and it actually lifts it actually lifts p. So, you see what is happening is I have x sub univ and I have this map p to X, then I have X tilde p tilde, this is a cover this is your covering given to you and then there is this point c x here which goes under p to x that is this point small x tilde here which goes also to the same x and what happens is that I am able to define this phi sub phi sub X tilde and this diagram commits. So, this phi sub X tilde is a lifting of p and it has a property that it sends c x to X tilde; if I start with the if for if for alpha I took the constant map at x then it will go down just to the constant it will it will just go what I get here is the constant path at x and its lift will be unique lift will be just the constant path at x tilde. So, you know constant path at x will go to x tilde. So, this is what happens under this map.

So, let me write that also note that phi X tilde is the unique lift of p to X tilde taking c x to X tilde because you know a covering map you know there are local homeomorphism has this uniqueness of lifting property. So, if you have 2 maps if you have any lifting that agree at a point then they will agree everywhere. So, this lifting is completely controlled by the fact that this c x this point goes to x tilde. So, it is completely determined by this X tilde.

Now, in addition suppose. So, this is what is called the universal property of the covering. So, what it does is the universal cover you give me any other cover the universal covering space is also a covering space for any other covering space this is also a covering map.

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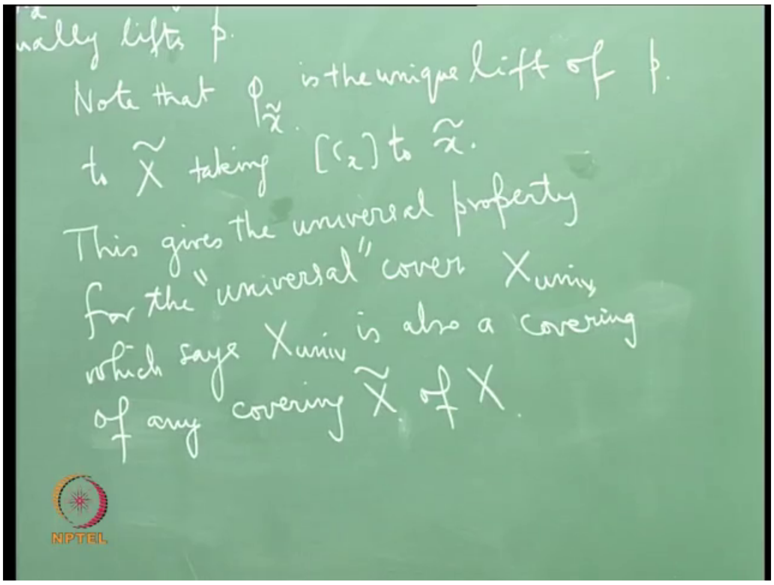

That is he that is this is the so called universal property that is the reason why it is called universal ok. So, this gives the universal property for the universal cover so. In fact, I should not put this in quotes I should rather put this in quotes x univ which says which says x univ is also the universal covering have it is also let me say first is also a covering a covering of any covering X tilde of x this is the universal property of the universal covering ok.

Now, now what I am going to do is I am going to say the following I am going to say that suppose in this whole construction that X tilde were also simply connected suppose X tilde were also simply connected, then you can define an inverse map and that will be continuous and that will give you a homeomorphism of this with that and that gives you the uniqueness of the universal covering. So, let me let me state that. So, that also essentially uses only the unique path lifting property ok.

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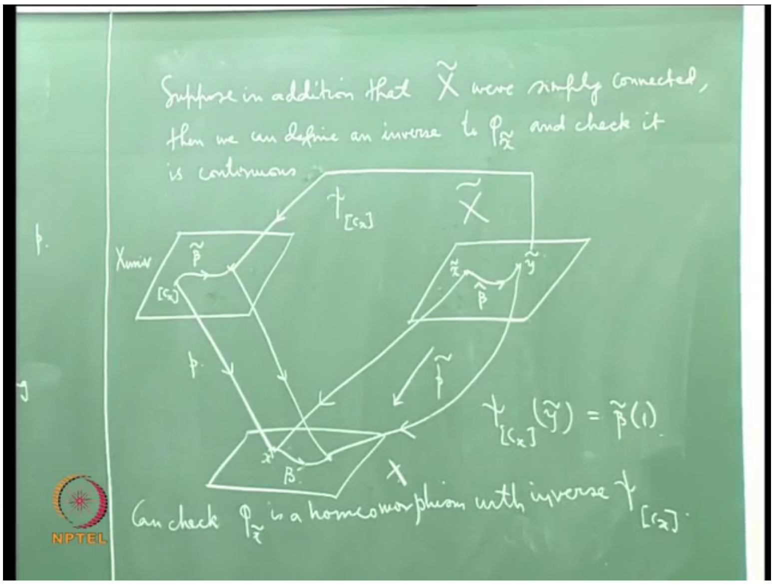

So, let me write that down suppose in addition that X tilde were simply connected, then we can define an inverse to phi X tilde and check it is continuous.

So, you see essentially the game I played to define this map p sub X tilde the same game I can play to define a map in this direction which will be an inverse namely what is that map see from x. So, here is X tilde and so, it is something like this. So, I will need to draw a diagram again. So, let me again draw diagram. So, here is X tilde and here is p tilde this is x and this is the universal covering x of univ and what I fixed here is the point c x which is which is lying over the point x under the map p and I have fixed a point there namely x small x tilde which is lying over which is again lying over the point x under the map p tilde.

Now, what I am going to do is I am going to define psi sub c x that is a map from here to here and the way I do it is follows give me a point y tilde here give me a point y tilde here what I will do is because X tilde has arc wise connected I simply choose a path; let me call this as I will choose a path here from X tilde to y tilde right and I push that path down I get a path here if I call this path as beta then this path is just the unique lift a beta starting at x tilde. So, this is this is beta hat alright and now what I will do is I will also take the unique lift of this beta to x univ. So, what I am going to get is I am going to get I am going to get this path here which you all know is beta tilde that is there I am going to get an end point and this is a map. So, this map is going to send y tilde to this point.

So, psi sub c x is going to send y tilde to beta tilde of one and you see. So, you see this point which is beta tilde of 1 that lies over beta of 1 and of course, y also lies over beta of 1. So, this is also true y also lies over beta of 1 and if I instead of beta hat if I chosen something else which is another path from X tilde to y tilde small x tilde to small y tilde it is anyway going to be homotopic to beta hat because X tilde is simply connected. And therefore, it is images here are going to be homotopic, and therefore the lifts here also going to be homotopic. So, this endpoint is going to be uniquely determined. So, the simply connectedness is just used to tell me that I can define this map properly.

Now, again a little bit of exercise will tell you that this map is continuous and it is an inverse to phi sub X tilde and therefore, phi sub X tilde will turn out to be a homeomorphism. So, what it will tell you is if X tilde to x is a covering space with X tilde simply connected then that has to be homeomorphic to the universal covering space. So, this gives you the statement that any covering of x with the property that the covering space is simply connected has to be isomorphic to this covering and it is isomo is not just a homeomorphism of these 2 spaces alone it is a homeomorphism that respects p projections. So, it is a homeom.

So, it is an isomorphism of covering spaces. So, it is an isomorphism from here to here which this followed by this is this projection and similarly phi phi sub X tilde going in this direction followed by this is this projection. So, can check can check phi sub X tilde is a homomorphism with inverse psi sub c x and so. So let me repeat this gives you the fact that any 2 covering spaces of x with which are simply connected or homeomorphic in a way that respects the covering projections and this is the so called uniqueness of the universal covering of a space.

So, we have also got the uniqueness of the universal covering now I need to go on and tell you why the fundamental group below is can be recognized as subgroup of homeomorphisms of the covering universal covering. So, let me see whether I have written that down properly.

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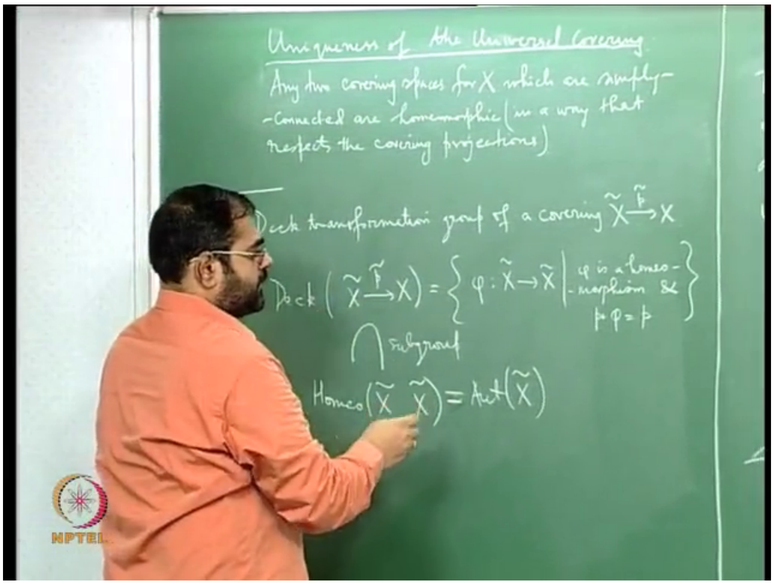

So, let me write one more line any 2 covering spaces for x which are simply connected are homeomorphic in a way that respects the covering projections. So, this is this is what you get and this is a unit. So, this is the so called uniqueness of the universal covering. So, this is the uniqueness of the universal covering what it tells you is that you can get a homeomorphism and the homeomorphism is unique if you fix a point in one and a point in and the other. So, we say it is unique up to unique isomorphism. So, if you want you can find many if you find if you have 2 simply connected covers of the same space you can get many homeomorphism, but if you want to fix the homeomorphism you have to fix a point here and it is image there then you have fixed it fine.

Now, what I need to do is I need to tell you that the fundamental group of x the fundamental group below sits as a sub group of automorphisms of the universal covering. So, I have to identify certain subgroup of automorphisms of the universal covering and these are the so called deck transformations. So, let me define the deck transformation group of a covering. So, what is this? So, is defined like this the deck. So, the notation that I am using is the following deck X tilde x p is the set of all phi from X tilde to x to X tilde such that phi is a homeomorphism and phi respects the projection namely first apply phi then apply p you get back p.

So, basically the diagram is like this. So, you can see these are of course, homeomorphism from X tilde to itself. So, this is this is certainly a subgroup of the homeomorphisms from X tilde to X tilde which is just automorphisms of X tilde after the automorphisms of X tilde as a topological space the self homeomorphisms from X tilde to X tilde this is a certain subgroup the what is the subgroup these are those which behave will with respect to the projection; and this actually means that these automorphisms they act on X tilde in the fiber direction.

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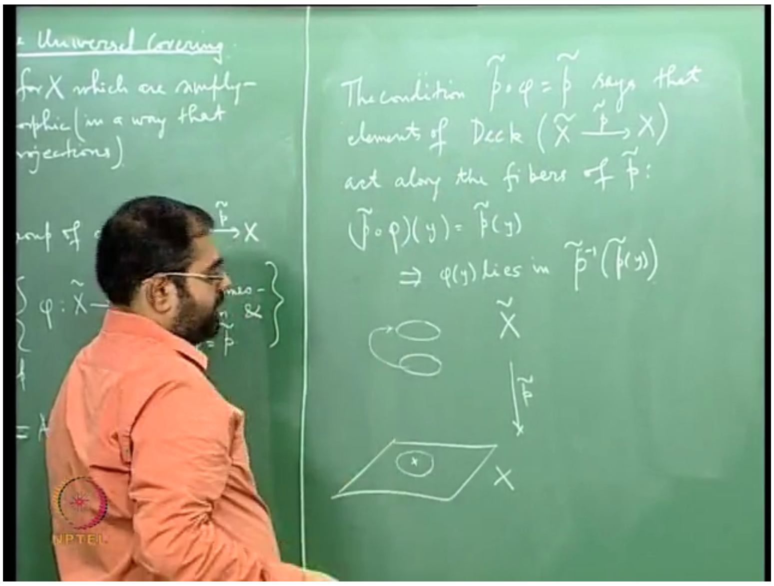

So, let me write that down the condition p circle phi is equal to p says that elements of the deck transformation group act along the fibres of p of namely; you see p circle phi of say a point y is equal to p of y. This actually means that you know phi of y lies in p inverse of y p inverse of p of y.

So, you can you can imagine what is happening. So, basically this is your x and this is your. So, of course, I have used p tilde everywhere. So, I should be careful about p tilde which should have been p tilde here because I have reserved p for the map that we have constructed. So, let me change it everywhere to p tilde. So, the situation is like this you give me a point and you give me a admissible neighbourhood the inverse image under p tilde will be several admits several neighbourhoods which are homeomorphically map by p tilde on to this.

And what this deck transformation will do will essentially it will move the this so called decks; you can think of them as various decks of a ship if you want or floors of a building and the action of each of these deck transformations is going to be only in the vertical direction. Now you can guess what I am going to do; I am just going to show that the fundamental group the; I am just going to show the fundamental group of x capital X based small x is canonically identifiable with this and that finishes it.

So, let me do that; it is not very difficult to do.

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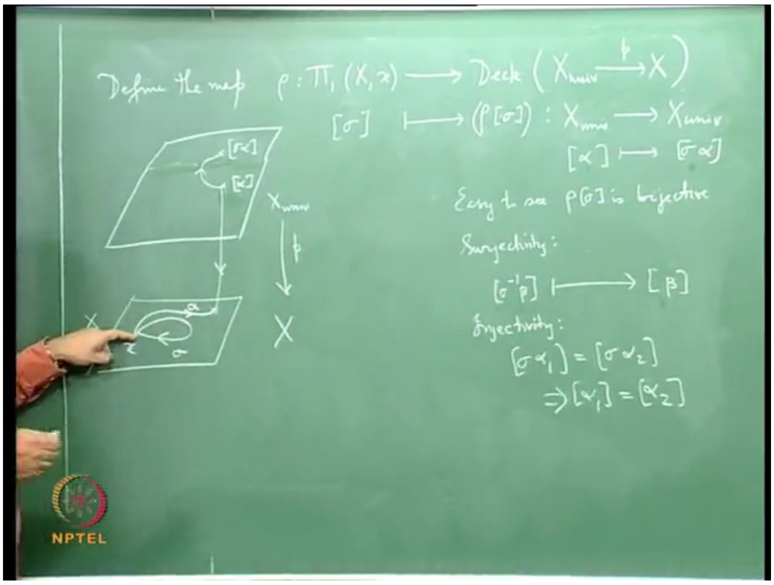

So, define the map. So, here is the map rho fundamental group x to fundamental group of capital X based small x to the deck transformation group of. So, I am now going to take the universal cover that we constructed I am going to find this map. So, of course, what I have said here is for a; this is a deck transformation group of a general covering I am now going to restrict to the case of the universal cover.

So, what is this map? So, it is the following. So, what I do is suppose I start with a sigma now sigma here is a loop base at x. So, my diagram is going to look like this. So, here is here small x and here is my sigma and what I am supposed to do is I have to give you a map from x sub univ into itself. So, I start with I start with a point alpha in x sub univ and I will tell you where it goes to under the image of this. So, this will go to rho of sigma I will define this as a deck transformation and so. So this should be a map from x univ into itself and what is this map I will send alpha to sigma alpha ok.

So, you see if I start with the point alpha that is going to give me a path here alright and then it is going to end somewhere and this point is going to be the point to which this alpha is going to be mapped under this under the under the covering projection. So, this is going to this and if I take; if I take sigma followed by alpha that is also a path starting at x and ending at alpha of one. So, it is going to be another point here. So, sigma alpha is also going to be here and therefore, you see your map is like this. So, it is a map that pushes along the fibre.

Now, 2 or 3 things can be seen very very quickly actually you can directly see that this map is bijective that is very very is easy to see easy to see to see a rho of sigma is bijective that is quite easy to see why is that true because you see surjectivity if I want to get hold of a beta then you know I have to just send it to I have to just take sigma inverse beta this will go to sigma sigma inverse beta which is beta. So, this is surjectivity and injectivity is obvious if sigma alpha 1 is equal to sigma alpha 2 then alpha 1 and alpha 2 are homotopic fixed end point. So, it is bijective there is nothing to its very very easy to check that, but in fact the point is that this map is a map of this form that is the whole point and that is the reason why it is a homeomorphism why is it a map of this form its.

So, if I write it down. So, let me rub this; let me draw another diagram.

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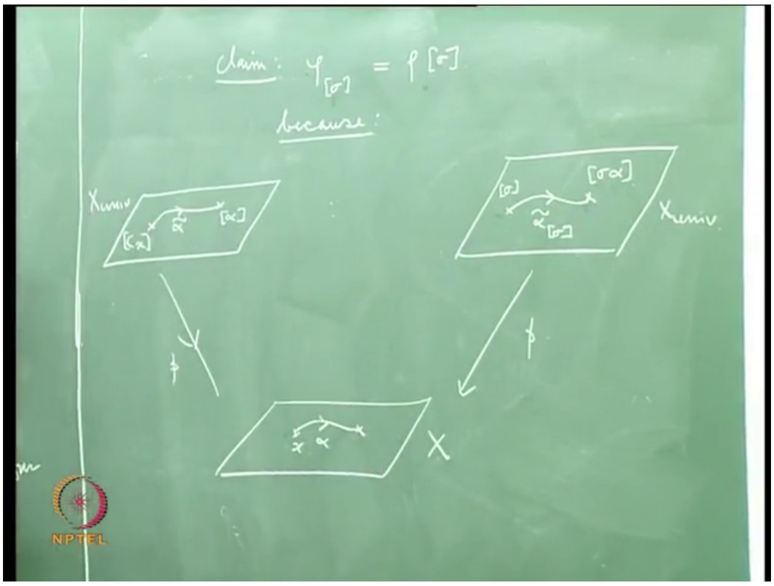

So, you see what is happening is the following I have my x here, I have my universal covering here and I have again my universal covering here, and these are of course again the covering projections. What I am fixing here is of course, over the point small x the point I am fixing is c x notice that c x goes to sigma ok.

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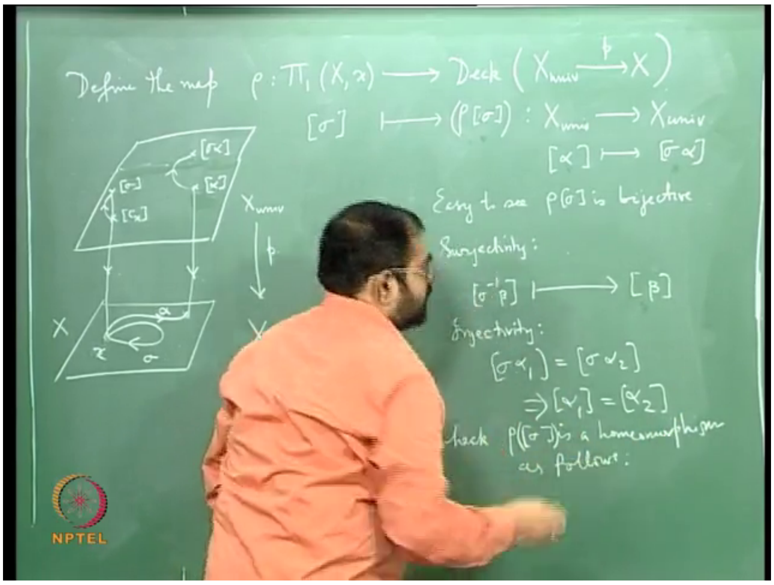

And notice also that c x and sigma are points over x see over x you have the point c x constant path at x homotopic class and sigma is loop.

So, that is the you see that is also a point here and both of them are lying above x and this multiplication by sigma acts on the fibre over x in the most natural way this is just multiplication by sigma on the left and in every over every fibre that is what is happening. So, you see choose this c x take x and then you take here the point a fix; I fix the point sigma which is a point over x. And I claim that if you take the corresponding map phi sub sigma I claim that this map is literally the same as rho of sigma I claim that these 2 are the same I claim that these 2 are the same. Of course, you know you have to do some work to check that rho of sigma is homeomorphism check un well. In fact, to check the rho of sigma is homeomorphism you can check that these 2 are equal and then use the fact that phi sub sigma is a homeomorphism ok.

So, let me write that down check rho of sigma is a homeomorphism is a homeomorphism as follows by this claim, but see why is this true if I take; if I take a point alpha here where will this go to under phi sub sigma where will it go to? So, well if you give me a point alpha, then this is a this represents the point the path alpha here, then I will get this thing here which is see this is the canonical lift of this alpha to the universal covering which is given by the shrinking homotopy and if I push it down I get alpha.

Now, I take the lift of this starting at sigma. So, I will get I will get a lift alpha tilde, but now this time starting at sigma this is a this sigma is another point over the over x another point in the fibre. So, I get and what is this endpoint this endpoint is precisely rho of sigma see this endpoint will turn out to be actually sigma alpha. So, you can see right and why this endpoint will turn out to be sigma alpha is again because of the shrinking homotopy construction. Therefore, these 2 maps are the same, but if you already verified that this is a homeomorphism then. So, is this; therefore, and of course, the way I have the way I have defined it; it is going to respect the projection.

So, it is going to be a homeomorphism of the universal covering which respects the projection. So, it is going to be a deck transformation. So, this map certainly lands inside here if I change sigma up to a homotopy nothing is going to change there is no problem if I change sigma by homotopy you can check. It is obvious that nothing is going to change.

So, this is a well defined map you can see that the identity element will go to the identity element here, because here is just multiplication on the left by sigma. So, if sigma was identity element, this will be just the identity map. So, the identity element of this group goes to the identity element of that group and you can easily verify that this rho is a group for morphism because it is just left multiplication literally. So this, the a group homeomorphism and it is easy to check that this is both injective as well as surjective. So, that can be checked very easily.

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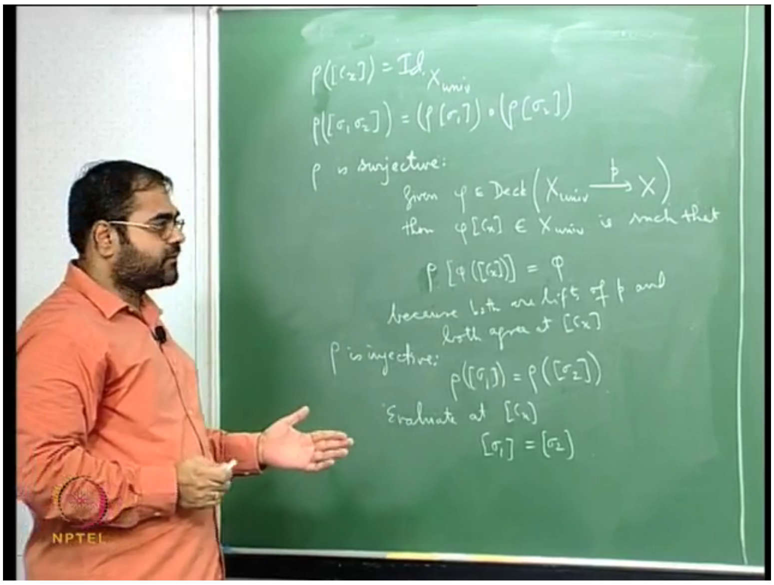

So, let me do that and then we are done. So, a rho sub; so what was it rho of rho of c x is identity map on x sub univ and rho of sigma 1 sigma 2. So, I should say multiplication in the group here is rho sigma 1 composed by with rho sigma 2 a rho is surjective how does one see that rho is surjective; it is very very easy to see because see suppose you give me given a deck transformation.

Let us say phi which is the deck transformation from x sub univ p to x then you see this deck transformation will take the constant path at x to a certain path. So, phi of c x is going to be a point of x sub univ such that you see rho of rho of phi of c x will be just phi because you see rho of see of rho of sigma which is phi sub sigma. So, rho of this for the left hand side is rho of this is just phi sub it is phi sub. Now I need see this these 2 are equal because they are both lifts of p and they both agree at c x because both are lifts of p and both agree at c x that is all.

So, rho is surjective and rho is injective namely you know if rho of sigma 1 is equal to rho of sigma 2 then again the same reason both of them will agree I mean you apply these to c x apply this to c x, you will get evaluate at c x; c x will go under this to sigma 1 and c x will go to sigma 2 homotopy class under this. So, you will get injective. So, the upshot of this is that the upshot of this is that the fundamental group below is naturally a sub group of automorphisms of the universal covering above and it is something that can be very explicitly seen.

So, I think with this from the next lecture onwards. I will move on, I will go back to Riemann surfaces and I will start deducing the all the fundamental results that I told you by analysing Mobius transformations.