An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-dimensional Tori and Elliptic Curves

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Lecture – 13
The Universal Covering as a Hausdorff Topological Space

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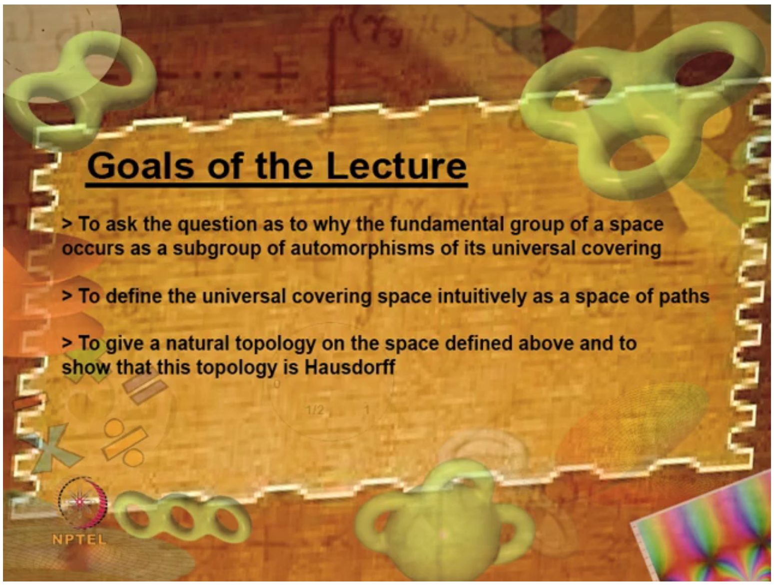

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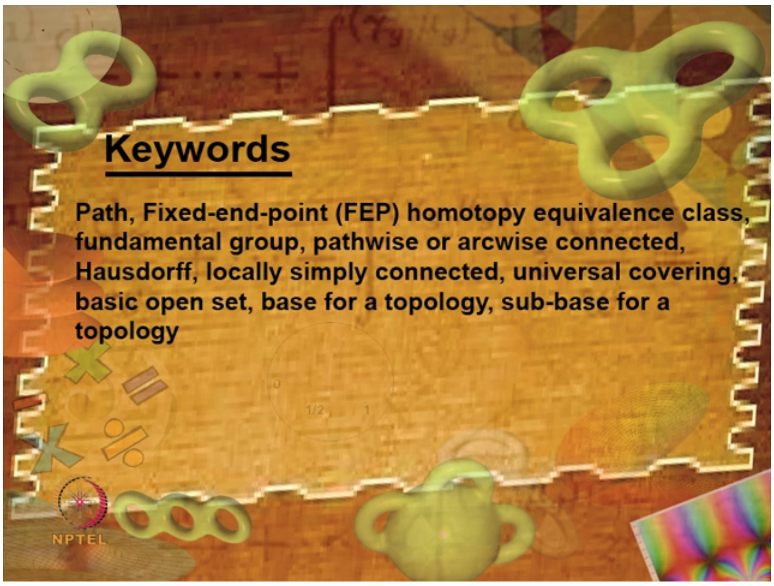

So let us continue with our discussion about covering spaces. So, you know we let me recall in the last lecture we looked at why the fiber over a point of the base space of the universal covering can be identified with the fundamental group base at that point. So, we got a picture like this.

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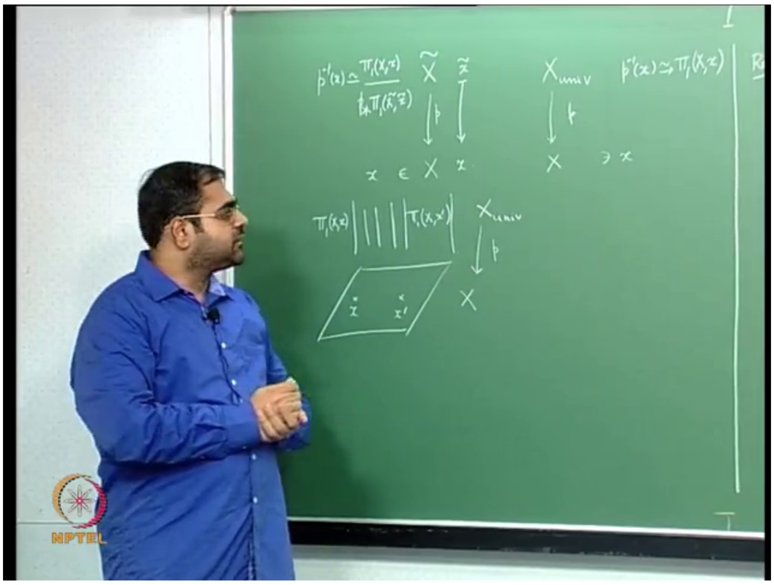

So, if x sub univ is the universal cover for the topological space X then I told you that given a point small x in capital X then the fiber p inverse of x namely the set of all points which are mapped to x under p is canonically which means naturally bijective to the fundamental group of capital X based at small x.

So in fact, we had a statement for a general cover, a general covering from which this followed. The namely if you had a general covering space a covering map if x tilde to x is a general covering map then you take a point small x in capital X, then the fiber p inverse of x, I said was it can be canonically identified with the co set space being the fundamental group of the space below at the based at the point below mod the image of the fundamental group above where x tilde was a point that going to x.

So, each fiber each fiber looked like this co set space and p lower star was the group homomorphism from the fundamental group above to the fundamental group below fixing a base point above which goes to a base point below and p lower star was an injective homomorphism. Therefore, p lower star of the fundamental group above was a subgroup of the fundamental group below and this co set space turned out to be canonically that is naturally identifiable with the fiber. And as a special case when x tilde is the universal covering then you know that for a universal covering the covering space is simply connected. So, the fundamental group is trivial and therefore, this term goes away this becomes just the identity subgroup and you get this.

So, we got a picture like this, we got a picture in this form. So, this was x and this x tilde on this X, now in this case the universal cover over X was a space which has got by you know for every point x I put a copy of the fundamental group based at small x. So, you take some other point x prime then I get the inverse image is the fundamental group at x prime. So, in this way the universal covering space is just all these fundamental groups put together as at least as a set. And in other words we say that this mapping is a vibration with fiber isomorphic to the fundamental group below and when I say fundamentally group below I do not worry about the base point because x is an arc wise connective space and the fundamental groups at two different points are isomorphic.

So, now, this explains why the fiber looks like the fundamental group below, in for a covering for a covering map, but inverse universal covering. Now we need to also look at another question the other question is the following.

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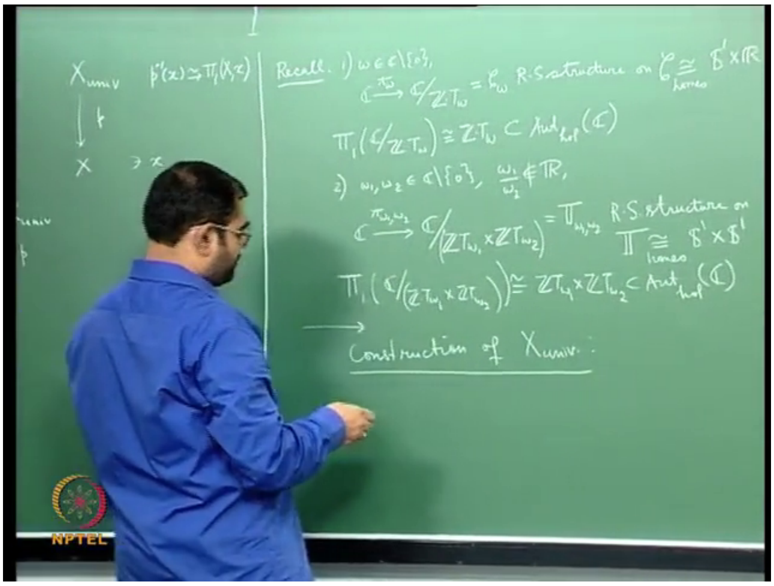

So, let me recall. So, let me draw line here. So, let me recall again there that you know we looked at these basic examples started with a non zero complex number and looked at the map from C to C mod the group of integer translations by this non zero complex number. And this map gave this map was into a quotient and this quotient inherited a Riemann surface structure and, so this was the Riemann surface structure C sub omega Riemann surface structure on the cylinder.

So, this is Riemann surface structure on the cylinder which is just which is which is homeomorphic to S 1 the circle cross R. And what we noted was that the first fundamental group of the cylinder and mainly the first fundamental group has got nothing to do with the Riemann surface structure it is just defined topologically. This first fundamental group is isomorphic to this group of translations by integer metals of omega which is a subgroup of automorphisms, holomorphic automorphisms of the covering space which is C this is the covering map this is the universal covering because C is simply connected and the fundamental group of the base is isomorphic to a sub group of holomorphic auto automorphisms above.

And similarly we had a similar situation for the torus. So, in the case of the torus complex torus we fixed two complex numbers and of course, assume that they are linearly independent over R, so the ratio is not real and we took the following map namely going modulo the integer the translation by integer multiples of these two complex numbers.

And this turned out to be a Riemann surface structure on the complex on the on the torus on the topological torus on the topological torus which is which I just called it as T which is homeomorphic to S 1 cross S 1. And of course, this map is say pi sub omega 1 comma omega 2 this also is a an example of a universal covering map and again in this case we find that the fundamental group of the topological space below is isomorphic to the this group which is just a group of the holomorphic automorphisms of the universal covering.

So, the question is the fundamental group of the base space in both cases can be identified as a subgroup of the automorphisms of the cover of the universal covering space. So, why does this happen we need to understand this. So, that is what we are trying to now understand and the way to do that or at least one way to do that is first to tell you how one can construct the universal covering space. So, of course, this argument will also help us to construct the universal covering space and eventually it will also explain why the fundamental group of the base space can be nicely identified as the automorphisms a sub group of automorphisms of the universal covering space. So, if let us start. So, construct let us give with the construction of the universal covering space.

So, as I have always told we are all we are assuming all our topological spaces to be. So, in particular x for example, we are assuming that the topological space the topological spaces are all housed of arc wise connected, locally arc wise connected and locally simply connected. So, that is a blanket assumption. So, what is one going to do? So, this is how we construct the universal covering space.

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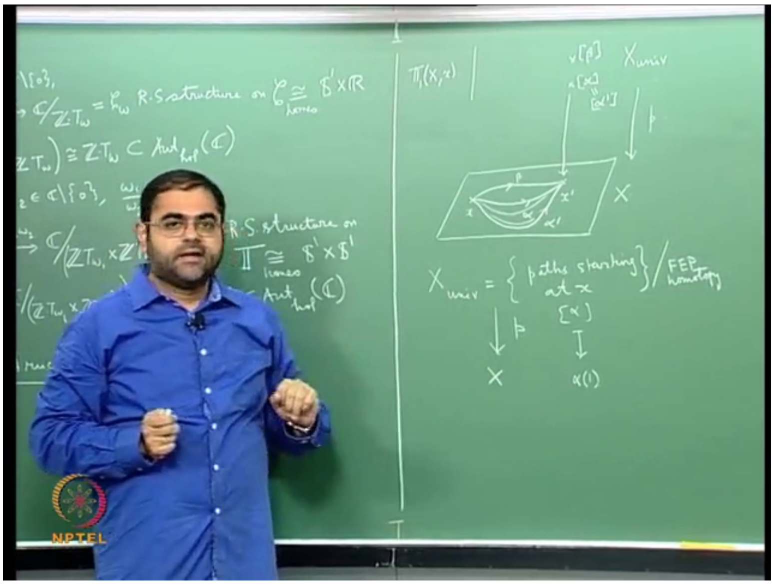

So, let me draw a diagram. So, it is very clear that you know. So, if this is X, I want this X the universal covering if you give me the point small x what I need on top is a copy of the fundamental group at half x based at x at least if you look at this situation.

So, basically we want the at least as a set we want the universal cover to satisfy this diagram namely or each point you have a copy of the fundamental group below based at that point below. So, it is very clear that over x I will have to put the fundamental group at x, but then if you take another point let me say x prime what is it that we are going to put on top. So, the answer to that is the following what one does is that one takes paths one takes paths alpha from x to x prime such paths always do exist because x is arc wise connected. So, I can take a path like this. And I can take the homotopic classes class of this path and put all those paths on top.

So, this is what is going to go to x prime and you see if x prime is equal to x what am I going to do I am going to just put going to put homotopic classes of paths starting at x at n and ending at x and therefore, I am going to get the fundamental code. So, there is a nice simple definition of the universal covering space at least to begin with as I said it is going to be just the set of all paths starting at x, used you have path starting at x and then you go modulo homotopic fixed endpoint homotopic.

So, you know if alpha and alpha prime are two paths such that you know alpha can be deformed continuously to alpha prime then they will give rise to the same point above in the fiber over x prime. And of course, you know what x prime is x prime is just alpha of one it will also be alpha prime of one it is the terminal point of alpha and alpha prime and of course, a beta which is not homotopic to alpha is going to give you a different point about. So, you can see that when x prime is x, I am going to simply get the fundamental group copy of the fundamental group above x. So, now, and what is going to be what is this map going to be. So, I have to define this covering map something that I should verify that finally, will turn out to be occurring map and the map is just a very natural map a from this set to this topological space and this map is simply you give me a path alpha and I just send it to the endpoint of the path alpha.

So, that corresponds to this diagram here. So, a point above lying over a point below is just homotopic class of a path from x to that point below. So, (Refer Time: 15:14) endpoint homotopic class right. So, this is my starting point. And now one to begin with first of all if you go back to this diagram you see that if I took any other point x prime what I would get in the fiber p inverse of x prime was something that naturally looked like the fundamental group based at x prime. So, we need to check whether that is going to happen there. See it is already happening for x because that is the way we started it, but it does not look that way from the definition, but one needs to do a little bit of work.

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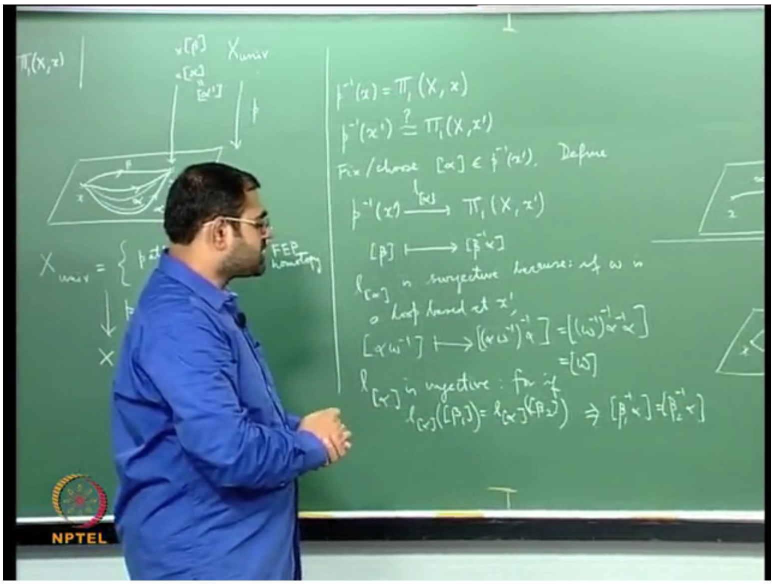

So, let me write this down see p inverse of x is certainly the fundamental group of x based at x small x, there is no doubt about it because this is going to be paths starting at x and also ending at x because of this definition. So, these are going to be loops at x modulo fixed endpoint homotopy. So, it is going to give me just the fundamental group.

Now, what about a point like x prime, what is p inverse of x prime, why should this look like the fundamental group based at x prime. Well for that let us, so why is this going to look like the fundamental group of capital X based at x prime. So, to understand this let me fix let me fix or choose you know an alpha in this let me choose a point alpha here all right. So, this is in p inverse of x prime, what does it mean? It means that the endpoint of alpha is x prime right. And what I am going to do I am going to define a map. So, define from p inverse of x to the fundamental group at x prime. So, this is how I am going to show that the fiber over x prime looks like the looks like a copy of the fundamental group over x prime what I am going to do is.

See give me any beta it give me any beta here I am just going to send it to let me look at my notes here. So, that I do not mess up the notation yes. So, I will define a map l sub alpha and this l sub alpha is just beta going to I think beta inverse alpha. So, you see try to understand this I have fixed this alpha and given any beta I am going to beta inverse alpha; that means, I am going to go like this and then I am going to go back by alpha that is going to give me a loop at x prime and therefore, it is homotopic class is going to give me an element of the fundamental group based at x prime.

Now, the claim is that this is a bijective map; the claim is that this is a bijective map and therefore, every fiber has been identified with the fundamental group based at the point below provided you just choose one the identification was based on choosing a point on the fiber. So, why is this map is injective and surjective we can see that very easily l alpha is a surjective because let me again draw another diagram here.

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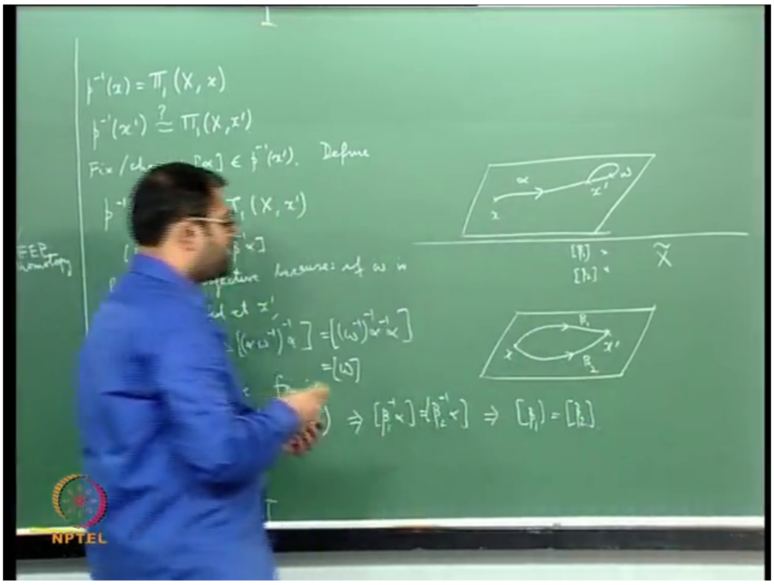

So, I have x prime, here is x this is the path alpha that I have chosen that I fixed the homotopic class of alpha has been fixed and you know I have to show that this is every this is the image of this map. So, if I start with an element here which is a loop based at x prime. If gamma if omega is a loop b is at x prime, you know my omega is going to look like this omega is a loop based on x prime. And well you know I have to find a beta such that beta inverse alpha is omega and that is, you know if you work it out you can see that I will have to just take, beta inverse alpha is omega what I need is a path I need an element over the fiber of x prime which means by definition a path from x to x prime and what is that path from x to x prime I am going to just take alpha followed by omega inverse. So, alpha followed by omega inverse, let us try this.

So, alpha is a path from x to x prime and omega inverse is also a path from x to x, x prime to x prime. So, if I compose them if I concatenate them I will certainly get a path from x to x prime and where does this go to this goes to by definition it goes to by this is map it goes to omega inverse it goes to alpha omega inverse whole inverse times of homotopic class of this. And if you write this down alpha omega inverse whole inverse is omega inverse the whole inverse alpha inverse and that is, this is going to be just omega inverse whole inverse alpha inverse alpha and you can see that this is just going to be omega. Because alpha inverse alpha is going to be, the alpha inverse followed by alpha is going to be homotopic to the constant path at x prime and omega followed by omega inverse whole universe is just omega, omega fall concatenated with the constant path x prime which is just going to be omega all right because the constant at path x prime is the you know it is the unit element in the fundamental group there based x prime.

So, therefore, it is surjective then I need to tell you that this is also injective the l sub alpha is injective for if. So, let us assume that l sub alpha of beta 1 is equal to l sub alpha of beta 2 suppose I assume this. So, again I have a diagram like this. So, let me draw another diagram here. So, I have two points above. So, this is, let me draw a line here. So, this is above this is in x tilde I have two points beta 1 and beta 2 and these two points are lying over the point x prime and what do we, what do each one of them correspond to? They correspond to homotopic the homotopic class of a path from x to x prime.

Namely, this is the path say this is the path beta 1 this is the path beta 2 and I have assumed that l sub alpha beta 1 is else of alpha beta 2. So, this by definition means that beta 1 inverse alpha is equal to beta 2 inverse alpha which means which tells you that beta 1 inverse alpha is homotopic to beta 2 inverse alpha and then you can operate on the right by alpha inverse and that will tell you that beta inverse is homotopic beta 1 inverse is homotopic to beta 2 inverse and that will tell you that beta 1 is homotopic would be beta 2. So, what this all this will tell you is that it will just tell you that beta 1 is equal to beta 2.

So, therefore, it happens that the map l alpha is certainly a bijective map and here since we are still in the same situation like this the only thing is that the fiber or x prime does not directly look like the fundamental group based at x prime you will have to it looks like that provided you fix a path from x to x prime. And now philosophically you can you can now see why this should be should happen this because you see if you give me a topologically space and give me two points and give me fundamental groups at those points the only way of saying that these two are isomorphic is by joining an arc. Once you join an arc then you get an isomorphism of this fundamental group with that fundamental group and, otherwise there is no way of connecting these two fundamental groups.

So, it is a necessary to connect x and x prime by a by a path or an arc and that we can do of course, because x is path arc wise connected and that is exactly what is happening here. All you need is a choice of a point here which is actually choosing an arc alpha from or a path alpha from x to x prime.

The moment you choose it you are able to identify the fiber with the fundamental group based at x prime. So, in some sense these athletes at least said theoretically a it is commensurately with this diagram, but we need to do more things we need to. So, what do we need to do? We need to make this into a topological space, we have to tell you, we have to tell what are the open sets here then we have to say that this map is a continuous map then we have to say that this topological space satisfies all the conditions that we want of topological spaces to satisfy in at least in our discussion namely you have to say that this is arc wise connected locally arc wise connected locally simply connected. Then we will have to tell that this is a covering map we will have to prove that there is a covering map then we will also have to say that this space is simply connected. Once you do all this then this becomes the universal covering so that is what I am going to do next right.

So, let me begin. So, let me make the following immediate observation. So, the first immediate observation is that this map p is certainly surjective because x is capital X is arc wise connected. So, p is surjective as capital X is arc wise connected.

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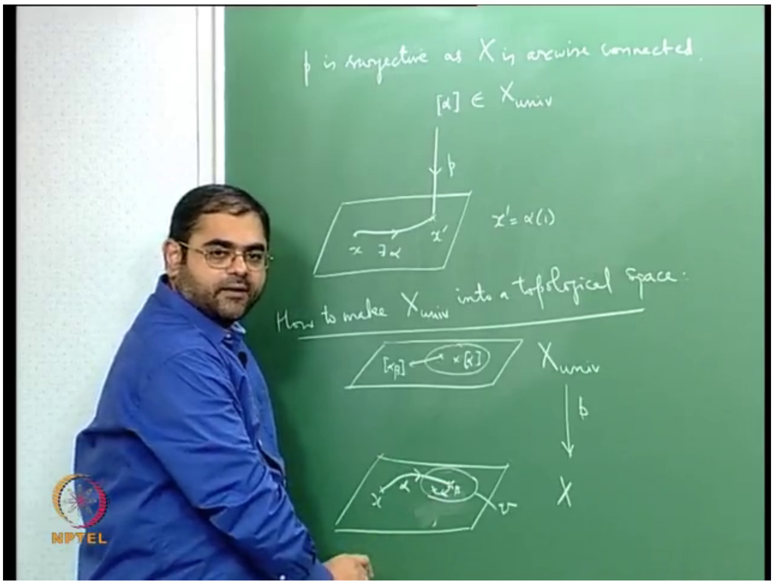

Why because well you give me you. So, here is x give me any other point x prime then that is certainly an a path alpha which connects x to x prime and then this homotopic equivalence class of this path alpha is an element of they said X sub univ which is mapped by p to x prime which is which is just x prime is just alpha of one the endpoint of alpha. So, it is very clear that the map is already surjective. I mean I think you should have noted notice that already.

Now, I need to I know I need to tell you about how to make this into a topological space. So, that is again how to make into a topological space. So, it is done in a very obvious way. You see, here is here is my this is my set above which I am trying to turn it over topological space and of course, in a way in which this map p becomes continuous. So, this map p is also to be thought of in this process. So, I have see, suppose I start with a point let us say alpha. So, this point is going to lie above the point x prime which is just alpha of one where alpha is a path from x to x prime and what would you imagine as a neighborhood of alpha you would imagine something like this and you would expect that to go down to you know the covering, covering map has to be an open map because it is a local homeomorphism. So, you would expect this to go down to something here. So, let me rub off this rub this off.

So, that let me draw it like this and let this be an open set U which contains alpha of 1 all right. And then if this map is continuous nearby points should go to nearby points. So, what is a nearby point here it is something that should go to a nearby point here and what is if there is a nearby point here you know what I could do is that I could join this if I can join this by a path beta from alpha of 1 to this point then that will give me the point above maybe this point is going to be alpha followed by beta. And by continuity you want nearby points to go to nearby points. So, based on this idea we define an open neighborhood of alpha in the following way. So, it is exactly as intuitive as you see in the diagram below.

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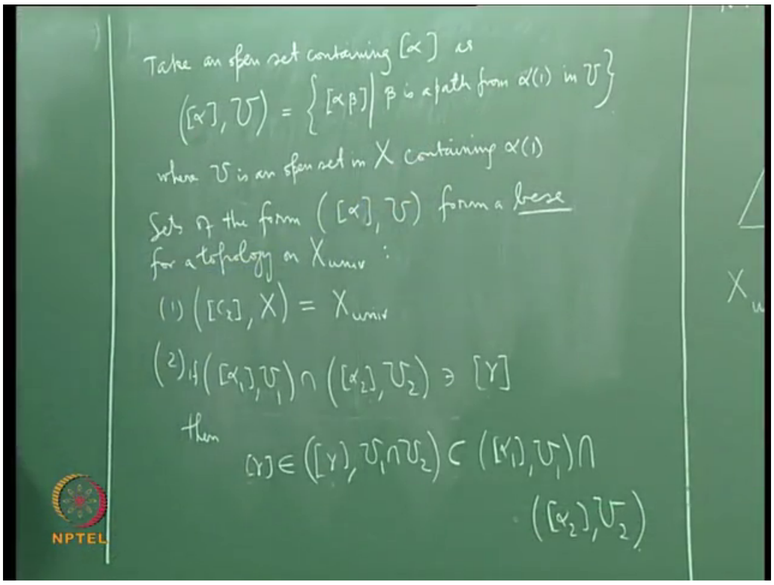

So, what you do is take an open set containing the point alpha as I will write it as a pair alpha comma U where U is an open set in capital X containing alpha of 1 which is the point above which alpha is lying. And what is this? See this should be all paths of that form namely this is all these paths of the form alpha beta where beta is a path from alpha 1 in U. So, beta is a path from alpha 1 in U all right.

So, what is happening is that, so long as I can connect a path in connect a point in U by a path from alpha 1 to that point then I take alpha followed by that path and put it in the set above and it is very clear that I am trying to here I am trying to reach all from alpha 1 I am trying to reach nearby points and therefore, the points I get above are going to be points close to alpha in a neighborhood of alpha. So, this is the intuitive way to define these open sets and once you define these open sets like this what happens.

So, what happens is they do not give you all the open sets, but they turn out to be basic open sets namely they form a base for the topology, for a topology and what does that mean; that means, that if you take finite unions of sets like this and then take arbitrary finite intersections of sets like this finite intersections of sets like this and then you take arbitrary unions these are going to give you all the open sets and sets of this form are certainly going to cover the this whole set.

So, let me write that down sets of the form alpha comma U form a base for a topology on x sub. So, why do they form a base? Well number one, you see if I take the path to be the constant path at x and I take the whole space then I will simply get every point in the universal in this universal set. So, if you take something like this then you are going to get everything there and the second thing is that if you take, if you have two such sets let us say alpha 1 U 1 and alpha 2 U 2, if you have two such sets and you have a point common to this say gamma then what happens is that this point gamma is contained it is belongs it belongs to the following set namely it is going to belong to. So, it is just going to belong to gamma U 1 intersection U 2 which is going to be contained in this intersection. So, this is what is going to happen.

So, what is happening is, I should put if here if two sets of this form intersect then you take any point in the intersection you can find again the set of this form containing that point which is in the intersection. So, this is the property of that says that these sets of basic open sets. So, sets form a basic open set, sets of certain type or set to form a base for the topology if you know you take two of them or say even finitely many of them and you take a point in the intersection then your should be able to find again a set of the same type in that intersection for finite intersection. So, I have done it for two intersections and you can do it for more than two, but finite intersections.

So, with these two properties it is easy to check that you can take the topology on x sub unit as the topologic given by open sets which are of the form you know arbitrary unions of finite intersections of sets of this type. So, let me write that down. I think maybe it is worthwhile to draw a diagram for this one.

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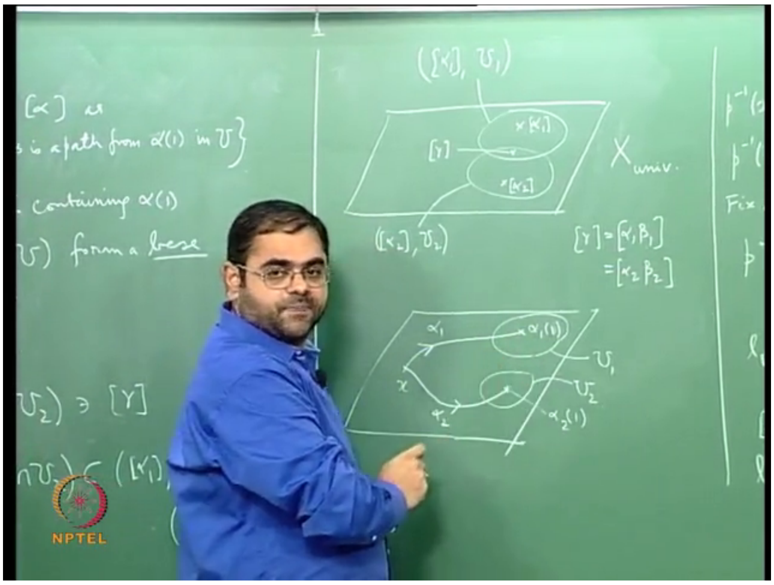

So, you see you have, above you have you have two points. So, you have two you have two neighborhoods of alpha 1 and alpha 2.

So, here is alpha 1 and then you have a neighborhood which is which is this neighborhood is alpha 1 comma U 1 and then I have another. So, that is alpha 2 here and this neighborhood is. So, this is this is all in X univ, so this is alpha 2 U 2 and how does it look like below. So, this alpha 1 corresponds to a point I to a path from x to alpha 1 of 0 alpha 1 of 1 and there is a there is this open set U 1 that I have chosen has a neighborhood of this point and then. So, this is alpha 1 and then you have you have another path alpha 2 and I am going to have well the endpoint is going to be alpha 2 of one and this is going the set is going to be just U 2.

And you know saying that there is a point in the intersection would mean that there is going to be some that there is some there is some gamma here. So, I should put a square bracket to signify homotopic equals class. So, this is my gamma here all right. So, saying that gamma is in this as well as this is the same as saying that you know gamma is by going by this definition. So, gamma is also alpha 1 beta 1 where beta 1 is a path from alpha of 1 and in U 1 and it is also equal to alpha 2 beta 2 where beta 2 is a path in U 2 starting at alpha 2 of 1. And saying that these two are the same at that these two equivalence homotopic equivalence classes are the same would tell you that the endpoints of these paths is the same because it is fixed at endpoint homotopy and that would mean that the endpoints of these have to be in the intersection of U and U 2.

So, the diagram does not really look like this. So, the diagram looks more like more like this. So, let me write that, let me draw one more diagram. So, diagram would look like this. So, you would have that diagram would actually look like this.

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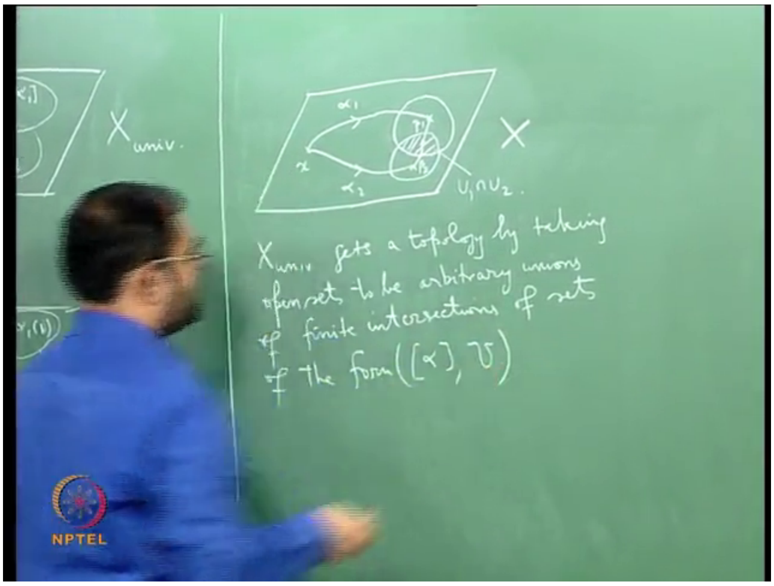

The diagram below will actually look like this, namely the diagram for X where you are going to have x, you are going to have alpha 1, you are going to have alpha 2 and then you know at the end point of our alpha 1 you are going to have a beta 1 and then you are going to have from the endpoint of alpha 2 you are going to have beta 2 and this alpha 1 followed by beta 1 is homotopic to 2 alpha 2 followed by beta 2 which is the which is what you can call as gamma if you want.

So, it means that this point is in the intersection of U 1 and U 2 all right. So, you know this point is in U 1 as well as in U 2. So, this diagram looks like this rather. So, so that is this common intersection. So, U 1 and U 2 have to intersect all right and then if U 1 and U 2 intersect then I can consider this set of this form. So, this is a path which starts at x and ends at that point which is the end point of beta 1 beta 2 which is lying in U 1 intersection U 2 and U 1 intersection U 2 is a non empty open neighborhood of that point. So, you get this set of this form and it is clear that a set of that this is certainly contained in this as well as that. So, this thing that shaded here maybe it a let me draw a slightly neat a diagram.

So, you know, this is alpha 1 this is alpha 2 and say this is beta 1 and this one is beta 2 and this is U 1 intersection U 2, and if you want you can call gamma to be either alpha 1 followed by beta 1 or alpha 2 followed by beta 2 or anything homotopic to that and take the equivalence class. So, this is x of course, and this is U 1 intersection U 2. So, the moral of the story is that you get a topology on this X sub univ.

So, X sub univ gets a topology by taking open sets to be arbitrary unions of finite intersections of sets of the form alpha comma U. So, that is how you make this into a topological space, but the idea is very very similar. You want to pick points close to in an open neighborhood of alpha you just do that by picking points close to the endpoint of alpha namely points in a neighborhood of an neighborhood of an of the endpoint of alpha.

So, now, that you have to topology, topology like this the first thing I need to tell you is that with this topology this map is at least continuous.

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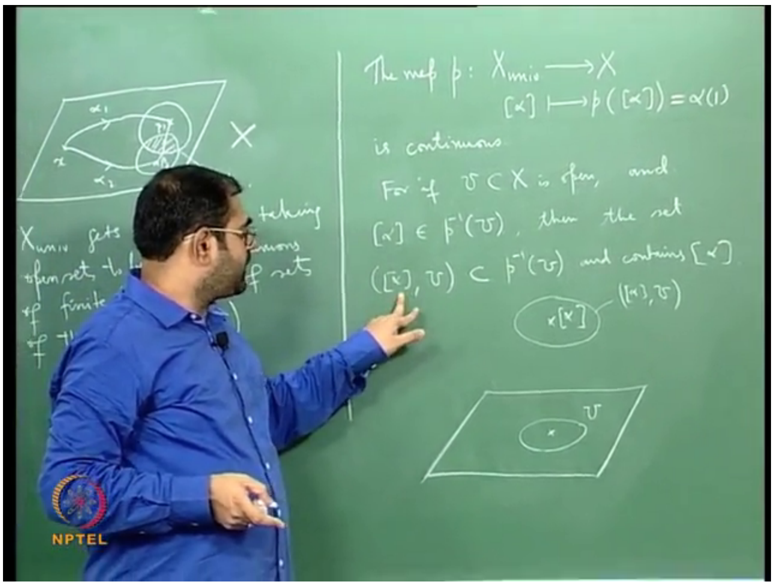

So, the map p from X union to X alpha going to, it is p of alpha is just alpha of one that is of a map is continuous and why is this continuous that is again quite easy to verify as follows. So, for I will have to just verify that that is continuous at each point. So, what I will do is I will start with the point and then I will take its image I will take an open neighborhood of that point and take the inverse image of that open neighborhood and show it is open. So in fact, what I will have to do is I have to check that the inverse image inverse images is open sets are open.

So, what I do, let me do this for if U in x is open and say alpha is in p inverse of U right. So, then the set alpha comma U is also going to be in p inverse of U and contains alpha. So, basically what is happening is that you know if you take an open set here U and you take an alpha above and, what do I have to do is I will have to show that p if U is an open set below I had to show p inverse U is open. So, I will have to show that every point of p inverse U is an interior point namely I have to show every point of p inverse U is surrounded by an open set which is contained in p inverse U.

So, what I do is take a point of p in p inverse U. So, that goes to a point here namely this point is just the end point of alpha. And then you know that I have this whole neighborhood above namely alpha comma U which is an which is an open set each of these are also open sets the basic open sets are also open sets because they correspond to. So, when I say take the open sets to be arbitrary unions of finite intersections. So, the finite intersection could be just one set that is the intersection of a family which contains only one element and the arbitrary union may be just union of a family which contains just one element.

So, each of these are also there in this collection. So, each of these are also open sets and clearly if you take a set like this then its image is going to go into U that will tell you that this is going to be in p inverse U all right and it contains the point alpha. So, alpha is surrounded by an open set an open neighborhood inside p inverse U and that is true for every point in p inverse U, so p inverse U is open. So, we have verified the inverse image of an open set is open. So, therefore, p is continuous.

So, that makes finally, this into a topological space and this into a continuous map. What we next need to do is to verify several properties of this topological space, namely we will have to show that it is you know housed off you have to show it is arc wise connected you have to show its locally arc wise connected locally simply connected and then you have to show it is also simply connected you have to show that this map is a covering map. So, maybe I can I let us go on to a try and tell you why this map is I mean why this space is housed off which is very very important to begin with.

So, I will try to explain that. So, my next claim is that this space is housed off.

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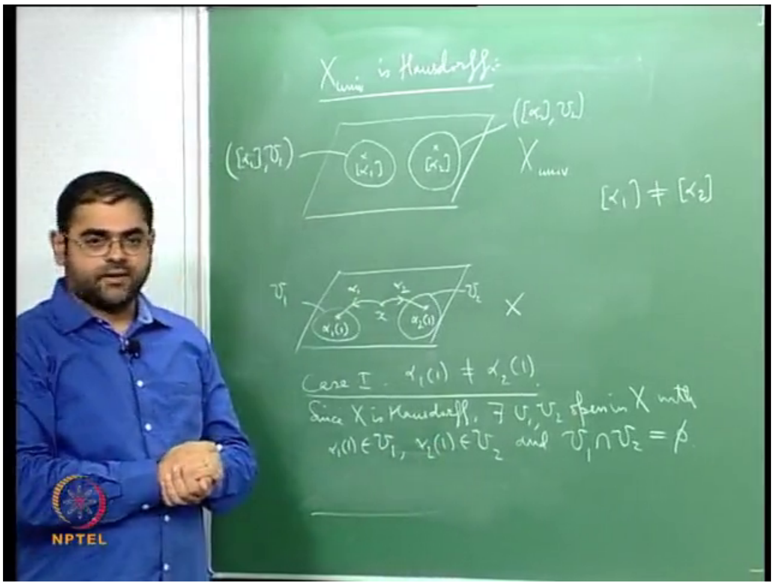

So, how do I show that this housed off? Well, what is the definition of space being housed off, you give me two distinct points then I can separate them it is joint open neighborhoods. So, here is my space about X sub univ and I have two points let us say alpha 1 and alpha 2 and they will lie up above their endpoints in x. So, here is a x the point I have fixed and you know alpha 1 is path to from x to alpha 1 of one which is the image of this point under p and well then I have another point path alpha 2 that will go into alpha 2 of 1 at will end at alpha 2 of 1 and that is the image of this entropy.

And suppose alpha 1 and alpha 2, there are two cases of when alpha and when alpha and alpha 2 have different endpoints and when they have a same end point. So, let us let us look at both the cases easily. So, I mean one by one case one is alpha 1 of 1 it is not equal to alpha 2 of 1 this is a very very easy case. And you see these are two points in x and x is housed off. So, there are two open sets, two open sets which an open set which contains is of an open set which contains that and which do not intersect. So, since x is housed off there exists U 1 U 2 open in x with you know alpha 1 of 1 in U 1 alpha 2 of 1 using U 2 and U 1 intersection U 2 being empty. So, namely the pictures going to look like this. So, I have U 1 here. So, this is U 1 and I have a U 2 here, this is U 2.

Then it is very clear that you know if I take this neighborhood above given by U by alpha 1 comma U 1 and this neighborhood here above by alpha 2 comma U 2 that these two that these two neighborhoods do not intersect. So, let me write the down.

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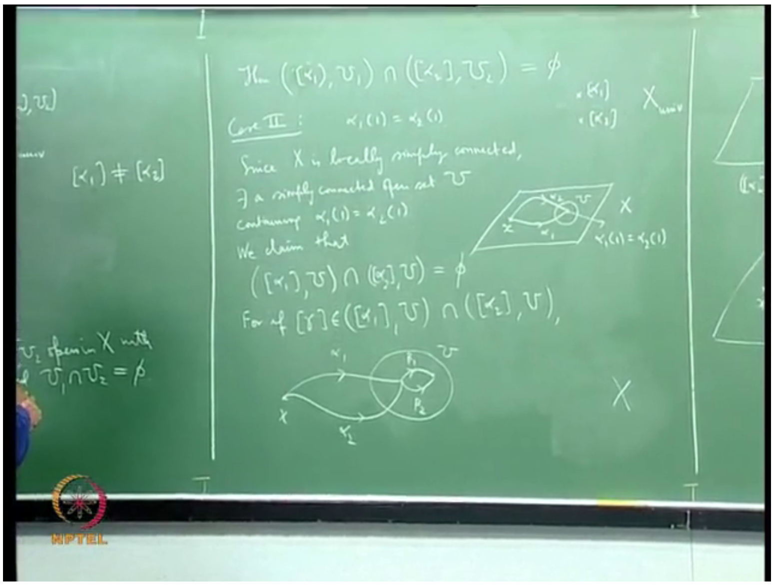

So, then U 1, so alpha 1 comma U 1 intersection alpha 2 comma U 2 are two open sets above they are neighborhoods of alpha homotopic class of alpha 1 and homotopic class of alpha 2 which do not intersect.

So, I will have to go on to the next case namely when which is slightly more trickier the case when both of these lie over the same point. So, alpha 1 of 1 is equal to alpha 2 of 1. So, situation is the diagram is like this what I have is I have two points alpha 1 above and alpha 2 above that lie over the same point below which is alpha 1 of 1 is equal to alpha 2 of 1 and, alpha 1 is also a path like this and alpha 2 is also path like this and x is a starting point. So, I have the situation.

So, then the question is what do I do. So, the question the point is that you see the space x. So, this is in X tilde this is in X sub univ and both of these points are going to this point which is the end point of alpha 1 as well as of alpha 2. The point there what we need to use now is the fact that x is you know locally simply connected. Since x is locally simply connected this every point of x has a simply connected neighborhood. So, what you do is you choose U to be a simply connected neighborhood of the point of this endpoint which is the same endpoint for alpha 1 and alpha 2 and the claim is that the corresponding sets above the same U are going to be disjoint.

Choose since, let me write that down since capital X is locally simply connected there exist a simply connected neighborhood open set U containing alpha 1 of 1 this is equal to alpha 2 of 1. Then we claim that the open sets alpha 1 comma U and alpha 2 comma U their intersection is empty. So, these are going to be a set separating open sets and the point is that well these are going to be two these are going to be two neighborhoods which are going to map homeomorphically on to U later on we will see that.

But for the moment the claim is that these two are, these two go into intersect and the proof for that, so this is alpha 2 this is alpha sub 2. So, the proof for that makes use of the fact that U is simply connected namely what you do is that you assume that there is a there is an intersection non trivial intersection. So, you get a point here and then you get a contradiction to the fact that alpha 1 and alpha 2 are distinct point. So, you see I am starting with alpha 1 not equal to alpha 2; that means, alpha 1 alpha alpha 2 are not homotopic. So, you can get this contradiction for if gamma is in the intersection alpha 1 U intersection alpha 2 U what happens, what does it what happens. So, a situation is like this maybe I can again draw a diagram here. So, I have, this is all happening in x. So, I have x here small x. So, I have I have alpha 1 I have alpha 2 and this is a common endpoint for alpha 1 and alpha 2 and then you know U is a simply connected neighborhood and what have I assumed I must I have assumed that gamma is both in this as well as that.

So that means, gamma is alpha 1 followed by a path beta 1 in U. So, there is a path beta 1 in U starting at this endpoint and going to some point and alpha 1 followed by beta 1 is gamma up to homotopy and then similarly the same thing happens here. So, there is another path beta 2 alpha 2 followed by beta 2 is also gamma up to homotopic all right. Now, you see this fact that this U is simply connected will tell you that any closed path in U can be continuously shrunk to a point. So, what it will tell you is that you know it will tell you therefore, that alpha 1 and alpha 2 are homotopic and; that means, that the homotopic class of alpha 1 will be equal to the homotopic class of alpha 2, but that is not, but that is not true. So, this can be this can be easily seen.

So, let me see whether I can write that down easily all I have to do is. So, alpha 1 followed by beta 1 is homotopic 2 alpha 2 followed by beta 2. So, I can write let me write that down.

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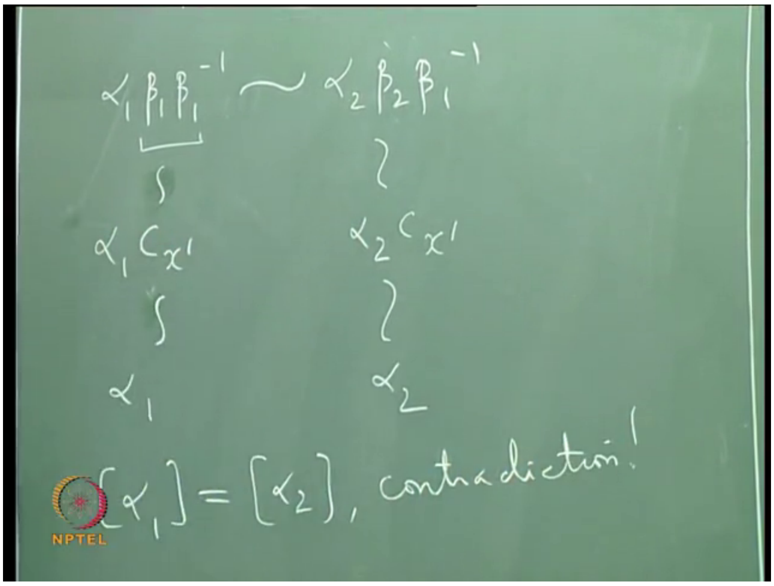

Alpha 1 followed by beta 1 is homotopic to alpha 2 followed by beta 2 and this homotopic class which we are calling as gamma and see now what I can do is that I can operate by beta 1 inverse on the right. So, what I will get is I will get alpha 1 beta 1 beta 1 inverse is going to be homotopic to alpha 2 beta 2 beta 1 inverse, but you see beta 1 beta 1 inverse, beta 1 followed by beta 1 inverse. So, let me call this point as x prime all right then beta 1, beta 1 inverse is going to be homotopic the constant path of x prime.

So, what I will get on this side this side is going to be just alpha 1 followed by the constant path that x prime and that is the same that is the same as, I should keep writing homotopy. So, which is the same as alpha 1 and what I have on this side is alpha 2 followed by what is beta 2 followed by beta 1 in inverse. So, you see beta 2 followed by beta 1 inverse is a loop at x prime. So, it is homotopic is the constant path that x prime because U is simply connected. So, what is going to happen is that I am going to get C x here also I am going to get C x prime and that is homotopic to alpha 2. So, the upshot of the story is that alpha 1 is equal to alpha 2 as homotopy, up to homotopy which is the contradiction.

So, this contradiction tells you that this intersection is indeed empty and therefore, the housed offness is proof.

So, in my next lecture let me try to give you the other properties of this topological space that make it into a universal covering for X. So, I will stop here.