Course Name: Essentials of Topology
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Welcome to Lecture 69 on Essentials of Topology.

In this lecture, we will discuss the well-known Tychonoff theorem. Specifically, we will see a proof idea for this theorem along with some concepts. We will discuss the detailed proof of this theorem in the next lecture.

Begin with what we have already seen during the study of the compactness of topological spaces, that the finite product of compact topological spaces is compact. Meaning is to say that if we are having compact topological spaces $(X_1, \mathcal{T}_1)$, $(X_2, \mathcal{T}_2)$, ..., $(X_n, \mathcal{T}_n)$, then $(X_1 \times X_2 \times \cdots \times X_n, \mathcal{T}_p)$ is compact, where $\mathcal{T}_p$ is the product topology. A natural question is: instead of taking these finite number of compact topological spaces, let us take an arbitrary family of compact topological spaces $\{(X_i, \mathcal{T}_i) : i \in I\}$. If we are taking the product $(\prod_{i\in I} X_i, \mathcal{T}_p)$ of these spaces, then the question is: whether $(\prod_{i\in I} X_i, \mathcal{T}_p)$ is compact. The answer is yes, and that is known as the Tychonoff theorem. Formally, what the Tychonoff theorem is, this theorem states that an arbitrary product of compact topological spaces is compact in the product topology. It seems simple, but the proof is not. In order to prove this theorem, we will use two concepts. One concept we have already studied is the concept of finite intersection property (FIP). We have already studied the characterization of compactness in terms of this property. The second concept that we require is the concept of Zorn's lemma.

Let us recall the concept of finite intersection property. We have already seen that a topological space $(X, \mathcal{T})$ is compact if and only if every collection $\mathcal{A}$ of closed sets in $(X, \mathcal{T})$ having FIP, the intersection, that is $\bigcap_{A\in\mathcal{A}} A$ is nonempty. Note that a collection $\mathcal{A}$ of subsets of X is said to have finite intersection property if for each finite sub-collection, say $\{A_1, A_2, ..., A_n\}$ of $\mathcal{A}$, $A_1 \cap A_2 \cap \ldots A_n \neq \emptyset$. This was the idea of finite intersection property, and by using this property, we have already characterized the compactness of a topological space. Now, the question is, how can we use this property to justify that the arbitrary product of compact spaces is compact, and what is the use of Zorn's Lemma there? Instead of giving formal proof here, just let us see what will be the proof idea, and accordingly, we will discuss the formal proof later on.

So, what exactly do we have with us that we are having a family of compact topological spaces $\{(X_i, \mathcal{T}_i) : i \in I\}$. Our motive is to justify that the product space $X = \prod_{i \in I} X_i$, along with the product topology, is compact. In order to justify it, let us assume a family $\mathcal{A}$ of closed subsets of X having FIP. In order to justify that this space is compact, what exactly do we have to justify that $\bigcap_{A \in \mathcal{A}} A$ is nonempty. The question is how to search for an element in $\bigcap_{A \in \mathcal{A}} A$. One idea that we may use is if we are taking any $i \in I$, let us use the notion of projection maps $\pi_i : X \to X_i$. By using these projection maps, let us take a collection $\{\pi_i(A) : A \in \mathcal{A}\}$ of subsets of X_i having finite intersection property. But it is to be noted that $\pi_i(A)$ may not be closed in X_i. So, what we can do that instead of taking $\{\pi_i(A) : A \in \mathcal{A}\}$, if we are choosing $\{\overline{\pi_i(A)} : A \in \mathcal{A}\}$. Then, this is a collection of closed subsets of X_i having FIP. Now, because $\{\overline{\pi_i(A)} : A \in \mathcal{A}\}$ is a family of closed subsets of X_i having FIP, and the space $(X_i, \mathcal{T}_i)$ is compact, we can conclude that $\bigcap_{A \in \mathcal{A}} \overline{\pi_i(A)} \neq \emptyset$. If this is the case, we can find an element $x_i \in \bigcap_{A \in \mathcal{A}} \overline{\pi_i(A)}$, and corresponding to these x_i, we can choose an element, let us take $x = (x_i)_{i \in I}$. Note that $x \in X = \prod_{i \in I} X_i$. But in order to justify that the product space is compact, our motive is to show that $\bigcap_{A \in \mathcal{A}} A$ is nonempty. So, the question is whether $x \in \bigcap_{A \in \mathcal{A}} A$. How do we make a guarantee for it? This is the main problem. Let us try to understand this by using this particular diagram.

If we are looking at Figure 1, what are we taking? We are taking this as a set X_1, this is X_2, and the box is representing the set $X_1 \times X_2$. Now, let us take a collection of subsets. So, this (triangle shape) is one member, and this (triangle shape) is another member of $\mathcal{A}$. Now, if we are trying to find out the projection of these subsets of $X_1 \times X_2$, so this is the projection of this particular set, and if we are taking this particular closed set, the projection is given here (line segment). Similarly, we want to find out the projection on X_2. So, for this set, the projection is here. Similarly, for this set, the projection is here (line segment). Note that their intersection is nonempty; they are satisfying FIP. Therefore, we can conclude that there will be an element that will be common to these. So, let us take x_1 as the element, and correspondingly, if we are taking an element x_2 here, this is an element in X_2. Now, if we try to find the corresponding element in this intersection, that is given by this point. For example, if this is our x_1, this is our x_2, so this point is precisely this ordered pair (x_1, x_2), and that lies in the intersection of these two sets.

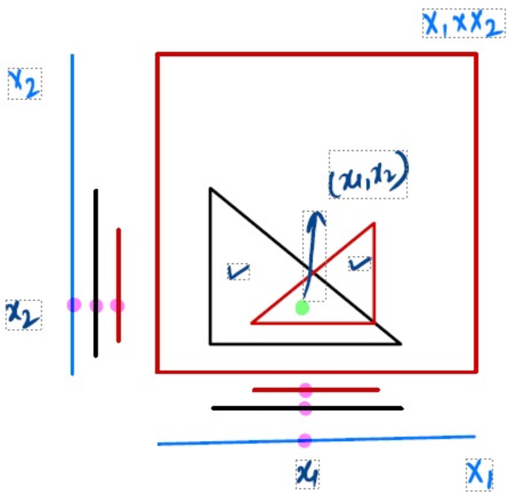

Figure 1: For element lies in the intersection

Similar to Figure 1, if we are looking at Figure 2, again, we are taking two nonempty sets X_1 and X_2, and the box represents the Cartesian product of $X_1 \times X_2$. Now, let us take the collection $\mathcal{A}$, which contains closed sets, that is, this one (represented as a triangle) as well as this one (represented as a triangle). Let us try to find out their projection on X_1. So, the projection of this closed set is given here (represented as a line segment), and the projection of this closed set is again here (represented as a line segment). Similarly, the projection of this set on X_2 is this one (represented as a line segment), and the projection of this closed set on X_2 is this one (represented as a line segment). Now, by using the methodology that we have used previously, let us take a common element, say x_1, and similarly, why not let us choose a common element x_2, here? If we are choosing the elements in this way, what will happen? Actually, the corresponding element, that is, (x_1, x_2), will be here (outside the common region). So, what are we getting? This (x_1, x_2), that is outside the intersection of these two closed sets. So, the main problem is: if we are constructing an element in this particular fashion, we cannot make a guarantee that this x will always be an element of intersection. So, what to do, and why such a problem arises? This is because of this arbitrary choice of element which we are choosing here. The question is how to fix this element inside the intersecting region. One of the ways to add some more elements in this family $\mathcal{A}$ so that the chosen element always lies in the intersecting region. Thus, what to do is if we add some more elements inside $\mathcal{A}$ and if we find their projection on X_1 and X_2. Now, if we are choosing elements in this region that will always lie in this intersecting area. But the question is how to choose such a family? The answer is that we can do it by choosing the maximal family of closed sets for which we will use the concept of Zorn's lemma.

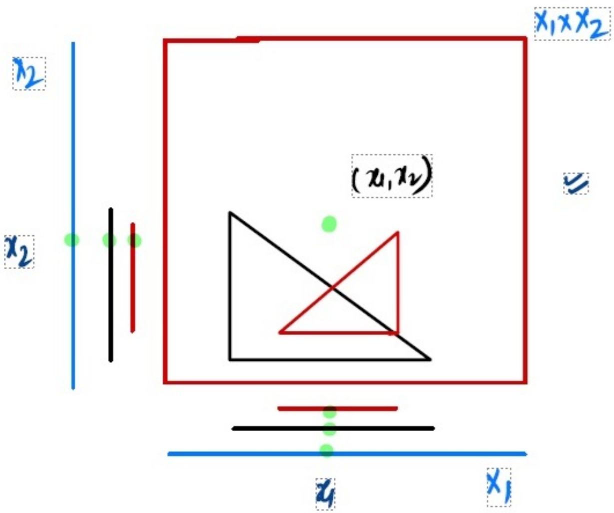

Figure 2: For element does not lie in the intersection

Let us have a look at Zorn's lemma and recall the concepts used therein. This lemma is well-known in the set theory. Let us see the statement of this lemma. Let $(X, \leq)$ be a partially ordered set in which every subset which is linearly ordered has an upper bound. Then $(X, \leq)$ has a maximal element. In this statement, we are using different concepts. For example, the first concept is a partially ordered set, the second concept we are using linearly ordered, we are using this concept, that is, the concept of upper bound, and finally, we are using the concept of the maximal element. Let us recall these concepts. Begin with the concept of a partial order on a set. A partial order on a set X is a binary relation (denoted by $\leq$) such that (i) $x \leq x$, for all $x \in X$ [reflexive], (ii) if $x \leq y$ and $y \leq x$, then $x = y$, for $x, y \in X$ [antisymmetric], and (iii) if $x \leq y$ and $y \leq z$, then $x \leq z$, for $x, y, z \in X$ [transitive]. The set X equipped with the partial order $\leq$ is called a partially ordered set (poset) and is denoted by $(X, \leq)$.

Let us give some examples. The well-known examples of poset are the set of natural numbers, the set of integers, the set of rationals, and the set of real numbers with the usual ordering. Even, if we are taking the set of natural numbers $\mathbb{N}$, and for $a, b \in \mathbb{N}$, let us define a relation, that is, $a \leq b$ if $a|b$. Then, we can see that this relation is also a partial order on $\mathbb{N}$. Thus $(\mathbb{N}, \leq)$ is also a poset. Even from set theory, let us take another example; that is, for a nonempty set X, take $P(X)$. For two elements $A, B \in P(X)$, let us define a binary relation, that is, $A \leq B$, provided $A \subseteq B$, that is the inclusion relation. Then this relation is a partial order relation, and therefore, $(P(X), \leq)$, is also a poset.

Moving ahead, let us see some more concepts. So, the first one is about the comparison between the elements of a poset. The elements x and y of a poset $(X, \leq)$ are comparable if either $x \leq y$ or $y \leq x$. A poset $(X, \leq)$ is called linearly ordered if every two elements are comparable. For example, if we are taking the posets $\mathbb{N}$, or $\mathbb{R}$ with usual ordering, in such posets, every two elements are comparable. Now, if we recall another example, where we have defined the partial order relation on the set of natural numbers in the fashion, that if we are taking two natural numbers a and b, we say that $a \leq b$ if $a|b$. Then if we are taking 3 and 6, as $3|6$,these are comparable. But instead of 6 if we are taking 5, note that 3 and 5 are not comparable. Accordingly, $(\mathbb{N}, \leq)$ is not linearly ordered. Further, if we are taking the poset $(X, \leq)$ and let us take $Y \subseteq X$, also take the restriction of $\leq$ on Y and denote this restriction by the same symbol, one can check that $(Y, \leq)$ is also a poset.

Moving ahead, let us see the concept of the upper bound, which is used in Zorn's lemma. What the concept is: Let $(X, \leq)$ be a poset. Then, an element $u \in X$ is said to be an upper bound of $Y \subseteq X$ if $y \leq u$, for all $y \in Y$. Let us see some of the examples. For example, if we are taking the poset $(\mathbb{N}, \leq)$, where this partial order is defined as: for $a, b \in \mathbb{N}$, $a \leq b$, if $a|b$. Now, if we are taking $Y = \{3, 4, 6, 9\}$. The question is, what will be an upper bound for Y. Can we say that 36 is an upper bound? The answer is yes. One can see that 36 is divisible by $3, 4, 6$, as well as 9. Even if we are taking the set of real numbers with the usual ordering on it, and $Y = [0, 1)$. Note that 1 is an upper bound of Y.

Finally, one more concept we have used in the statement of Zorn's lemma is the concept of maximal element. What is it? Let $(X, \leq)$ be a poset. Then an element $m \in X$ is said to be maximal if $x \in X$ and $m \leq x$ implies $m = x$. We have already studied the concept of maximality in different courses of mathematics. Now, let us take an example. Let X be a set having at least two elements and $Z = P(X) - \{\emptyset, X\}$. Now, for $A, B \in Z$, let $A \leq B$ if $A \subseteq B$. Then $(Z, \leq)$ is a poset. One can justify that for $x \in X$, $X - \{x\}$ is a maximal element of Z. The question is how? The answer is, why not let us take $A = X - \{x\}$, and let us take $B \in Z$ such that $A \leq B$. Then, we can show that $A = B$. In case if $A \neq B$, there exists $z \in B$ such that $z \notin A$. But it is to be noted that what A is? A is nothing but $X - \{x\}$. It means that $z = x$. But in this case, we can conclude that $B = X$. It is to be noted that $X \notin Z$. Therefore, $A = B$. Hence, elements of the form $X - \{x\}$ are maximal.

These are the references.

That's all from this lecture. Thank you.