Course Name: Essentials of Topology
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Welcome to Lecture 67 on Essentials of Topology.

In the previous lecture, we discussed a well-known result in topology called the Urysohn lemma. In this lecture, we will see an interesting application of this lemma for proving the Tietze extension theorem, which is a deep theorem in topology. In order to prove this theorem, we will require a few concepts from analysis, too. Accordingly, first, we will recall some concepts from the analysis, and thereafter, we will prove this theorem formally in the next lecture.

Begin with, let us see the statement of this theorem, which we have to discuss in this lecture and in the next lecture, too. The statement of the theorem is given here: Let $(X, \mathcal{T})$ be a normal space, $A \subseteq X$ be closed, and $f : A \to [a, b]$ be a continuous function. Then there exists a continuous function $g : X \to [a, b]$ such that $g(x) = f(x)$, for all $x \in A$. In order to prove this theorem, what do we require? We require the concept of Urysohn lemma, which we have already discussed. Also, we require the concept of uniform convergence. When we talk about the concept of uniform convergence, we require the uniform limit theorem along with one more well-known concept from analysis, known as the Weierstrass M-test. So, we are going to recall these concepts, and after that, by using the Urysohn lemma, we will establish another result, and by using that result, we will prove this Tietze extension theorem.

So, let us begin with the concept of uniform convergence. What this concept is: Let $f_n : X \to \mathbb{R}$ be a sequence of functions from the set X to $\mathbb{R}$. Then the sequence (f_n) converges uniformly to a function $f : X \to \mathbb{R}$, if for given $\epsilon > 0$, there exists an integer N such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and for all $x \in X$. This definition is well-known in analysis. Also, we know that whenever we are talking about this N, this N only depends on ϵ, not on x. As this is a well-known concept in analysis, we are not discussing the concept of uniform convergence much. The question arises: if we take the sequence (f_n) as a sequence of continuous functions, then if this sequence converges to

f, is this f also continuous? The answer is yes, and the result is known as the uniform limit theorem. Let us see this theorem in a topological framework, which we have to use here.

This theorem is stated as: Let $(X, \mathcal{T})$ be a topological space and $f_n : X \to \mathbb{R}$ be a sequence of continuous functions. If (f_n) converges uniformly to f, then f is continuous. It is to be noted that $f : (X, \mathcal{T}) \to (\mathbb{R}, \mathcal{T}_e)$. If we want to justify that f is continuous, why not let us take $G \in \mathcal{T}_e$, and show that its inverse image under f is $\mathcal{T}$-open, that is, $f^{-1}(G) \in \mathcal{T}$. It is to be noted that if $f^{-1}(G) = \emptyset$, obviously, f is continuous. In case $f^{-1}(G) \neq \emptyset$, in order to justify that $f^{-1}(G) \in \mathcal{T}$, let us show that this $f^{-1}(G)$ is the $\mathcal{T}$-neighborhood of each of its points. For which, let us take $x \in f^{-1}(G)$ and show that there exists a $\mathcal{T}$-open set, say H, such that $x \in H \subseteq f^{-1}(G)$. Now, the question is, how do we find this H? In order to find such H, let us try to use the uniform convergence of the sequence alongwith the openness of set G. Note that $G \in \mathcal{T}_e$. If $x \in f^{-1}(G)$, we can conclude that $f(x) \in G$. Therefore, there exists $\epsilon > 0$ such that $f(x) \in (f(x) - \epsilon, f(x) + \epsilon) \subseteq G$. Now, if we are using the uniform convergence of the sequence (f_n), we can say that for ϵ, or $\epsilon/3 > 0$, there exists a positive integer N such that $|f_n(x) - f(x)| < \epsilon/3, \forall x \in X$ and $\forall n \geq N$. It is to be noted that from here that we can construct an interval, that is, $(f_n(x) - \epsilon/3, f_n(x) + \epsilon/3)$. Note that this contains $f(x)$, and this is going to help us search for H. So, let us take $f_n^{-1}((f_n(x) - \epsilon/3, f_n(x) + \epsilon/3))$ as H. Then, the first conclusion, which we can draw, is that H is $\mathcal{T}$-open as it is the inverse image of a $\mathcal{T}_e$-open set. At the same time, it is to be noted that the function f_n is continuous. Further, $x \in H$. Why? Note that $f_n(x)$ is a member of $(f_n(x) - \epsilon/3, f_n(x) + \epsilon/3)$.

Finally, we have to justify that $H \subseteq f^{-1}(G)$. In order to justify it, let us take $y \in H$ and try to justify that $y \in f^{-1}(G)$. Now, if $y \in H$, then $y \in f_n^{-1}((f_n(x) - \epsilon/3, f_n(x) + \epsilon/3))$. From here, we can conclude that $f_n(y) \in (f_n(x) - \epsilon/3, f_n(x) + \epsilon/3)$, or that $|f_n(y) - f_n(x)| < \epsilon/3$. Now, our motive is to justify that $y \in f^{-1}(G)$. It is to be noted that G contains an open interval $(f(x) - \epsilon, f(x) + \epsilon)$. So, why not let us try to find $|f(y) - f(x)|$. Note that $|f(y) - f(x)| = |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)|$. Now, by using uniform convergence, we have seen that $|f_n(x) - f(x)| < \epsilon/3$, for all $x \in X$. Therefore, it holds for y too. So, we can conclude that $|f(y) - f(x)| < \epsilon$, or $f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$. At the same time, it is to be noted that

$f(y) \in (f(x) - \epsilon, f(x) + \epsilon) \subseteq G$. So, what we achieved is that $f(y) \in G$, or $y \in f^{-1}(G)$. Thus, $H \subseteq f^{-1}(G)$, or that $f^{-1}(G)$ is a neighborhood of each of its points, or $f^{-1}(G)$ is $\mathcal{T}$-open, or the function f is continuous. That's proof of the uniform limit theorem.

One more result we have to use is known as the Weierstrass M-test. We are just stating that here. The test is stated as: Given a sequence of functions $f_n : X \to \mathbb{R}$, let $s_n(x) = \Sigma_{i=1}^{n} f_i(x)$. If $|f_i(x)| \leq M_i$, for all $x \in X$ and for all i, and if the series ΣM_i converges, then the sequence (s_n) converges uniformly to a function s. We will see the example when we use this test to prove the Tietze extension theorem.

Moving ahead, let us prove a lemma that will play a key role in proving the Tietze extension theorem. The lemma is stated as: Let $(X, \mathcal{T})$ be a normal space, $A \subseteq X$ be closed, and $f : A \to [-r, r]$ be a continuous function. Then there exists a continuous function $g : X \to \mathbb{R}$ such that $|g(x)| \leq \frac{1}{3}r$, for all $x \in X$; and $|g(a) - f(a)| \leq \frac{2}{3}r$, for all $a \in A$. In order to justify this lemma, what exactly will we do? Actually, we will divide this closed interval $[-r, r]$ into three equal parts. Now, let $B = f^{-1}([-r, \frac{-r}{3}])$ and $C = f^{-1}([\frac{r}{3}, r])$. It is clear that B and C are disjoint. Further, B and C are the inverse images of closed intervals. So, what can we conclude that B and C are closed subsets of A. Note that $A \subseteq X$, is also closed. So, B and C are closed subsets of X. Further, as we have already discussed that B and C are disjoint, we can use the well-known Urysohn lemma. By using the Urysohn lemma, there exists a continuous function $g : X \to [\frac{-r}{3}, \frac{r}{3}]$ such that $g(b) = \frac{-r}{3}$, for all $b \in B$ and $g(c) = \frac{r}{3}$, for all $c \in C$. From the definition itself, we can conclude that $|g(x)| \leq \frac{1}{3}r$. What do we have to justify next? We have to justify that $|g(a) - f(a)| \leq \frac{2}{3}r$, for all $a \in A$. Now, if $a \in A$, then $a \in B$, or $a \in C$, or $a \notin B$ and $a \notin C$. For $a \in B$, we have already justified that $g(a) = \frac{-r}{3}$. The question is, what about $f(a)$? Note that B is nothing but the inverse image of $[-r, \frac{-r}{3}]$ under f. Thus, $-r \leq f(a) \leq \frac{-r}{3}$. In any case, we can conclude that $g(a)$ and $f(a)$, both are members of $[-r, \frac{-r}{3}]$, and therefore $|g(a) - f(a)| \leq \frac{2}{3}r$. If $a \in C$, we have already seen that if $c \in C$, $g(c) = \frac{r}{3}$. So, we can conclude that $g(a) = \frac{r}{3}$. Further, if we are looking at the behavior of this function f on any element of C, that will be governed by the fact that C is the inverse image of $[\frac{r}{3}, r]$ under f. Thus, $\frac{r}{3} \leq f(a) \leq r$, or that $g(a), f(a) \in [\frac{r}{3}, r]$. Therefore, it is clear that $|g(a) - f(a)| \leq \frac{2}{3}r$. Finally, if $a \notin B \cup C$, then we can conclude that $g(a)$ and $f(a)$ both are the members of $[\frac{-r}{3}, \frac{r}{3}]$. Why? Because we have already seen that $|g(x)| \leq \frac{r}{3}$, and if a is neither a member of B nor C, it is clear that $f(a) \in [\frac{-r}{3}, \frac{r}{3}]$. Therefore, we can conclude that $|g(a) - f(a)| \leq \frac{2}{3}r$.

Thus, in any case, for all $a \in A$, $|g(a) - f(a)| \leq \frac{2}{3}r$. That's the proof of this lemma. In the next lecture, we will see a formal proof of the Tietze extension theorem.

These are the references.

That's all from this lecture. Thank you.