Course Name: Essentials of Topology
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Week: 11
Lecture: 02

Welcome to Lecture 62 on Essentials of Topology.

In this lecture, we will also continue the study of the concept of separation axioms. Begin with what we have seen in the previous lecture. We have already seen the concept of T_0-spaces as well as T_1-spaces. Let us have a look at T_1-spaces. What have we seen in the case of T_1-spaces? If we take a topological space $(X, \mathcal{T})$, let us take two distinct elements $x, y \in X$. Our motive was to find two open sets; one contains x but not y, and another contains y but not x. If we take these open sets, something like G and H, is this G intersection H an empty set, or are G and H disjoint? We have not considered this in the case of T_1-spaces, but we are going to consider it now. So, begin with the definition of T_2-spaces or Hausdorff spaces, which we have already seen during the study of the notion of compactness. A topological space $(X, \mathcal{T})$ is Hausdorff if for every pair of distinct elements $x, y \in X$, there exist two open sets G and H such that $x \in G$, $y \in H$, and $G \cap H = \emptyset$. From the definition, it is clear that every T_2-space is a T_1-space, or every Hausdorff space is a T_1-space. The question is, what about its converse? We can see that the converse is not necessarily true.

Let us take some of the examples.

- Discrete topological spaces are Hausdorff.
- $\mathbb{R}$ with standard topology is Hausdorff.
- $\mathbb{R}$ with cofinite topology is not Hausdorff.
- Sierpinski space is not Hausdorff.

Note that $(\mathbb{R}, \mathcal{T}_{cf})$ is a T_1-space but not a T_2-space, that is a T_1-space need not necessarily be a T_2-space. Further, let us take one more example of a Hausdorff space, i.e, metrizable space. Let $(X, \mathcal{T}_d)$ be a metrizable space, where d is the metric. Then for $x, y \in X$, where $x \neq y$, $d(x, y) > 0$. Why not let us take

$d(x, y) = r$. Also, let us take $G = B(x, r/3)$ and $H = B(y, r/3)$. Then $x \in G$, $y \in H$, and $G \cap H = \emptyset$.

Moving ahead, we have seen one result regarding Hausdorff spaces. The result was: if the topological space $(X, \mathcal{T})$ is Hausdorff, then every single point subset of X is closed. This is similar to the result in T_1-spaces. In the case of T_1-spaces, we have justified that the converse of this result is also true. But the question is, what about in the case of Hausdorff spaces? The answer is that the converse of this result is not necessarily true. For example, if we are taking $(\mathbb{R}, \mathcal{T}_{cf})$, for all $x \in \mathbb{R}$, $\{x\}$ is closed. Why? Because this is a finite set. But we have already seen that this space is not a Hausdorff space.

Moving ahead, let us see some more results for T_2-spaces similar to T_1-spaces. The continuous image of a Hausdorff space is not necessarily Hausdorff. Why? The answer is, let us take $f : (\mathbb{R}, \mathcal{T}_e) \to (\mathbb{R}, \mathcal{T}_{cf})$ such that $f(x) = x, x \in \mathbb{R}$. It is to be noted that this function is a continuous function. Also, we have seen that $(\mathbb{R}, \mathcal{T}_e)$ is Hausdorff, while $(\mathbb{R}, \mathcal{T}_{cf})$ is not a Hausdorff space. Thus, the continuous image of a Hausdorff space is not necessarily Hausdorff. Still, we can justify that being Hausdorff is a topological property. How do you justify it? For the justification, let us take a homeomorphism $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ and $(X, \mathcal{T})$ as a Hausdorff space. Now, let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Note that f is a homeomorphism. So, we can use the concept of a surjective function. Thus, by using the surjectivity, we can say that there exist two elements $x_1, x_2 \in X, x_1 \neq x_2$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. So, we can conclude from here that there exist two open sets, that is, $G, H \in \mathcal{T}$, and this is possible because $(X, \mathcal{T})$ is Hausdorff such that $x_1 \in G$, $x_2 \in H$ and $G \cap H = \emptyset$. If $x_1 \in G$, we can conclude that $f(x_1) \in f(G)$. Also, $f(x_2) \in f(H)$, or we can say that $y_1 \in f(G)$ and $y_2 \in f(H)$. Now, being homeomorphism, f is open, too. Therefore, what about $f(G)$ and $f(H)$? These will be members of $\mathcal{T}'$; that is, these are $\mathcal{T}'$-open sets. The question is, whether these are disjoint, too. The answer is yes because f being injective, $f(G \cap H) = f(G) \cap f(H)$, and therefore $f(G) \cap f(H) = \emptyset$.

Moving ahead, let us see another result: a subspace of a Hausdorff space is Hausdorff. Similar result, we have stated in the case of T_1-spaces, but we have not proved there. Let us see proof here. In order to justify it, let us take a Hausdorff topological space $(X, \mathcal{T})$ and a subset $Y \subseteq X$. We have to justify that $(Y, \mathcal{T}_Y)$ is Hausdorff. In order to justify that $(Y, \mathcal{T}_Y)$ is Hausdorff, let us take two distinct elements $y_1, y_2 \in Y$. Because Y is a subset of X, $y_1, y_2 \in X$. As $(X, \mathcal{T})$ is Hausdorff, there exist $\mathcal{T}$-open sets G and H such that $y_1 \in G$, $y_2 \in H$ and $G \cap H = \emptyset$. From here, we can write that $y_1 \in Y \cap G$, $y_2 \in Y \cap H$, and $Y \cap (G \cap H) = \emptyset$. If we are taking $Y \cap G = G'$, and $Y \cap H = H'$, then $y_1 \in G'$, $y_2 \in H'$, and $Y \cap (G \cap H) = G' \cap H' = \emptyset$. It is to be noted that G' and H' are $\mathcal{T}_Y$-open. Therefore, $(Y, \mathcal{T})$ is a Hausdorff space.

Moving ahead, let us see the result related to the product topology. Similar result we have also seen in the case of T_1-spaces, but we have not proved there. The result states that if we are taking two Hausdorff spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$, the product space, that is, $X_1 \times X_2$ with product topology, is also Hausdorff. In order to prove it, let us take two distinct elements $x, y \in X_1 \times X_2$. So, what are we taking? We are taking $x = (x_1, x_2) \in X_1 \times X_2$. Also, we are taking another element $y = (y_1, y_2) \in X_1 \times X_2$. Note that $x \neq y$. It is to be noted that if $x \neq y$, it means that either $x_1 \neq y_1$ or $x_2 \neq y_2$. So, we can take any of the cases. We are assuming that $x_1 \neq y_1$. Now, it is to be noted that $x_1, y_1 \in X_1$ and $(X_1, \mathcal{T}_1)$ is Hausdorff. So, we can say that there are $\mathcal{T}_1$-open sets; let us take G and H such that $x_1 \in G$, $y_1 \in H$, and $G_1 \cap H_1 = \emptyset$. Now, let us use the notion of projection maps. Note that $\pi_1^{-1}(G) = G \times X_2$. Also, $\pi_1^{-1}(H) = H \times X_2$? We also know that these are $\mathcal{T}$-open sets. Why? Because projection maps are continuous. Thus, what we can conclude from here is that $x = (x_1, x_2) \in G \times X_2$, $y = (y_1, y_2) \in H \times X_2$. The question is: what about the intersection, that is, $(G \times X_2) \cap (H \times X_2)$. Note that it will always be an empty set. Why? Because G and H are disjoint sets. Thus, what have we justified? We have justified that for two distinct elements of $X_1 \times X_2$, there exist two disjoint open sets: $G \times X_2$ and $H \times X_2$ such that $G \times X_2$ is containing x and $H \times X_2$ is containing y. Hence, the product space is Hausdorff. This result can be generalized for arbitrary products of Hausdorff spaces.

Moving ahead, let us see an interesting characterization of Hausdorff spaces. The characterization is stated here: A topological space $(X, \mathcal{T})$ is Hausdorff if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ is a closed subset of $X \times X$. Obviously, $X \times X$ is endowed with the product topology. In order to justify it, let us assume that the space $(X, \mathcal{T})$ is Hausdorff. If we want to prove that the diagonal is a closed subset of $X \times X$, the best way is, try to show that its complement $(X \times X) - \Delta$ is an open subset of $X \times X$. Now, if we want to justify that $(X \times X) - \Delta$ is open, let us show that this is a neighborhood of each of its elements. So, why not let us take an element of this set, that is, let us take $(x, y) \in (X \times X) - \Delta$. From here, it is clear that $(x, y) \notin \Delta$, therefore $x \neq y$. Now, if $x \neq y$, what are x and y? These are elements of X. What is $(X, \mathcal{T})$? This space is Hausdorff. So, we can conclude that there exist two open sets; let us take them, G and H, such that $x \in G$, $y \in H$, and $G \cap H = \emptyset$, or that $(x, y) \in G \times H$ and $(G \times H) \cap \Delta = \emptyset$. Why? Because G and H are disjoint. Thus, from here, we can conclude that $G \times H \subseteq (X \times X) - \Delta$. Also, $(x, y) \in G \times H$. Thus, it is clear that $(X \times X) - \Delta$ is a neighborhood of (x, y), that is, it is the neighborhood of each of its elements. Therefore, $(X \times X) - \Delta$ is an open set. Hence Δ is a closed set.

Let us prove the converse of this result. For this, let us assume that the diagonal $\Delta = \{(x, x) : x \in X\}$ is a closed subset of $X \times X$. In order to prove that $(X, \mathcal{T})$ is Hausdorff, let us take two distinct elements $x, y \in X$. Then $(x, y) \notin \Delta$, or that $(x, y) \in (X \times X) - \Delta$. But at the same time, it is to be noted that Δ is a closed set. Therefore, $(X \times X) - \Delta$ is an open set. If $(X \times X) - \Delta$ is an open set, there exists $G \times H \subseteq X \times X$, open in product space, such that $(x, y) \in G \times H \subseteq (X \times X) - \Delta$. From here, we can deduce that $x \in G$ and $y \in H$. It is to be noted that $G, H \in \mathcal{T}$. So, we have shown the existence of open sets: one contains x, and the other contains y. The question is only to justify whether $G \cap H = \emptyset$. This is followed from the fact that $(G \times H) \cap \Delta = \emptyset$. Therefore, $(X, \mathcal{T})$ is a Hausdorff space.

Moving ahead, let us see another result. This result can be simply derived from the previous one. The result is stated as: Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a continuous function, where $(Y, \mathcal{T}')$ is a Hausdorff space. Then, the graph of f is a closed subset of $X \times Y$ (endowed with the product topology). Note that for a function $f : X \to Y$, the graph of f (denoted by $g(f)$) is given by $g(f) = \{(x, f(x)) : x \in X\}$. Our motive is to justify that $g(f)$ is a closed set. In order to justify that this set is closed, let us use the continuity of f. Now, let us construct another function $g : X \times Y \to Y \times Y$ such that $g((x, y)) = (f(x), y)$, for all $(x, y) \in X \times Y$. So, what this g is? We can say that this is nothing but the function f along with the identity function, i.e., f is sending x to $f(x)$, and we are taking the identity function on Y. Because f is continuous, we can conclude that g is also continuous. If this is continuous, let us use the diagonal $\Delta = \{(y, y); y \in Y\}$, which is a subset of $Y \times Y$. Now,

$g^{-1}(\Delta) = \{(x,y) \in X \times Y : g((x,y)) \in \Delta\}$, or $g^{-1}(\Delta) = \{(x,y) \in X \times Y : (f(x),y) \in \Delta\} = \{(x,y) \in X \times Y : f(x) = y\} = \{(x, f(x)) : x \in X\} = g(f)$. Now, the important thing is that g is a continuous function, and what is this diagonal? This is a closed subset of $Y \times Y$. Therefore, its inverse image is a closed subset of $X \times Y$. Because $g^{-1}(\Delta) = g(f)$, therefore, $g(f)$ is a closed subset of $X \times Y$.

These are the references.

That's all from this lecture. Thank you.