Course Name: Essentials of Topology
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Week: 09
Lecture: 05

Welcome to Lecture 53 on Essentials of Topology.

In this lecture, we will discuss the concept of compactness of subsets of $\mathbb{R}$ along with subsets of $\mathbb{R}^n$. Begin with what we have already seen if we are taking Euclidean topology on the set of real numbers, and if we are taking open intervals or semi-open intervals, we have seen a family of such intervals. These intervals are not compact. We have seen that the closed and bounded interval $[a, b]$ is compact. We have not yet seen proof of it. So, here, we will show that $[a, b]$ is compact. What specifically will we show? First, we will show that the closed interval $[0, 1]$ is compact. As we know, two closed intervals are homeomorphic. Therefore, $[0, 1]$ is homeomorphic to $[a, b]$, and if $[0, 1]$ is compact, then $[a, b]$ will be compact. Thus, we will show that $[0, 1]$ is a compact subset of $\mathbb{R}$, where $\mathbb{R}$ is endowed with the standard topology or Euclidean topology. Here, our proof will be based on the concept of connectedness. Let us see the proof of this result.

As we have to show that $[0, 1]$ is a compact subset of $\mathbb{R}$. Let us begin with an open cover of $[0, 1]$. So, what are we taking? Let us take this $\mathcal{C} = \{G_i : i \in I\}$ as an open cover of $[0, 1]$. If this is an open cover of $[0, 1]$, what does it mean? It means that $[0, 1] \subseteq \cup\{G_i : i \in I\}$. Now, if we are taking any element of this closed interval, let us take $x \in [0, 1]$. So, what will happen? There exists some G_i in this cover, that is, $\mathcal{C}$, such that $x \in G_i$, but it is to be noted what this G_i is. This is open, and if this is open, what will happen? There exists an interval; let us take that interval as I_x. It is to be noted that this interval is open in this subspace, that is, closed interval $[0, 1]$, such that $x \in I_x \subseteq G_i$. Having this information with us, let us construct a subset of $[0, 1]$. So, what are we going to construct? We are going to construct a subset $S \subseteq [0, 1]$, and this subset will help us to use the concept of connectedness. So, what this subset S is? Actually, S is as under:

$$S = \{z \in [0, 1] : [0, z] \text{ can be covered by a finite number of sets } I_x\}.$$

It means that if $z \in S$, $[0, z] \subseteq I_{x_1} \cup I_{x_2} \cup ... \cup I_{x_k}$, for some elements $x_1, x_2, ..., x_k$.

It is to be noted that $S \neq \emptyset$. Why? Because at least 0 will always be an element of this S. Note that this 0 belongs to this singleton set $\{0\}$, which is nothing but $[0, 0]$. So, S cannot be empty. Now, let us take an element $x \in S$ and also take $y \in I_x$. Let us see what will happen. It is to be noted that this y is an element of I_x. What is I_x? This is an interval containing x. It means that x and y both are members of I_x. Also, we are assuming that $x \leq y$. Because x and y are members of I_x, we can conclude that $[x, y] \subseteq I_x$. As we are taking $x \in S$ and if $x \in S$, it means that $[0, x] \subseteq I_{x_1} \cup I_{x_2} \cup ... \cup I_{x_k}$, for some elements $x_1, x_2, ..., x_k$. Now, if we are looking for $[0, y]$, what we can write from here is that this $[0, y] \subseteq I_{x_1} \cup I_{x_2} \cup ... \cup I_{x_k} \cup I_x$. It means that $[0, y]$ is covered by a finite number of sets of the form, that is, I_{x_i}, and if this is the case, we can conclude that $y \in S$. That is, if we are taking $x \in S$, then this $I_x \subseteq S$. From here, we can conclude that $I_x \cap S = I_x$. The question is, if x is not an element of S, can we deduce that $I_x \cap S$ will always be an empty set? It is a simple one, and just think about it.

If these are the cases, what can we do? Let us write $S = \{I_x : x \in S\}$, and we can also write the complement of this set S as $[0, 1] - S = \{I_x : x \notin S\}$. Thus, S is clopen in $[0, 1]$. But it is to be noted here, what S is? S is non-empty. But we know that intervals are always connected, so $[0, 1]$ is connected. If this interval is connected, and S is a clopen subset of $[0, 1]$, which is non-empty, what can we conclude? We can conclude that $S = [0, 1]$. Now, if $S = [0, 1]$, we can say that $[0, 1]$ is contained in the sets or union of sets of this form, that is, $[0, 1] \subseteq I_{x_1} \cup I_{x_2} \cup ... \cup I_{x_n}$. But as we have seen, these I_{x_i} are contained in some G_i, so what we can write from here is that this $[0, 1] \subseteq G_{i_1} \cup G_{i_2} \cup ... \cup G_{i_n}$. So, what have we obtained? We obtained $\mathcal{C}' = \{G_{i_1}, G_{i_2}, ..., G_{i_n}\}$, this is a subcover of $\mathcal{C}$. Thus, what have we shown? We have shown that if we are beginning with an arbitrary open cover of this closed interval $[0, 1]$, that open cover has a finite subcover, and hence this set, that is, the closed interval $[0, 1]$, is a compact subset of $\mathbb{R}$.

Moving ahead, and now what can we justify? We can justify that this closed interval $[a, b]$ is a compact subset of $\mathbb{R}$, where $\mathbb{R}$ is endowed with the standard topology. This is because of the reason which we have already discussed that $[0, 1]$ is homeomorphic to $[a, b]$. By using the compactness of $[a, b]$, what can we deduce? We can deduce that if we are taking closed intervals $[a_1, b_1]$, $[a_2, b_2], ...,$ $[a_n, b_n]$, that is, we are taking a finite number of closed and bounded intervals in $\mathbb{R}$, then their product, that is, $[a_1, b_1] \times .... \times [a_n, b_n]$, this is a compact subset of $\mathbb{R}^n$. How is this possible? What have we shown? We have shown that the intervals $[a_1, b_1], [a_2, b_2], ..., [a_n, b_n]$ are compact. What have we studied? We have studied that the finite product of compact subsets is compact. Therefore, $[a_1, b_1] \times .... \times [a_n, b_n]$ is compact.

Moving ahead, let us discuss the compactness of an arbitrary subset of $\mathbb{R}$; that is, if we are taking a subset $A \subseteq \mathbb{R}$, what will be the nature of it? So, what is the statement that is given here that every compact subset of $\mathbb{R}$ is bounded? In order to justify it, let us take $A \subseteq \mathbb{R}$. This is given that this A is compact. What we have to justify. We have to show that this A is bounded. We will prove the result by contradiction. For this, let us assume that A is unbounded. Now, use the compactness of A, and for using it, let us take a collection, that is, $\mathcal{C} = \{(-n, n) : n \in \mathbb{N}\}$. This is an open cover of A. If this is an open cover and A is compact, what can we write by using the compactness of A? We can write $A \subseteq (-n_1, n_1) \cup (-n_2, n_2) \cup ... \cup (-n_k, n_k)$. But it is to be noted that we have taken A as an unbounded set. So, there exists an $m \in A$, and what is this m? m is greater than the maximum of these $n_1, n_2, .., n_k$. Meaning is, the cover $\mathcal{C}$ has no finite subcover because m cannot be an element of $(-n_1, n_2) \cup (-n_2, n_2) \cup ... \cup (-n_k, n_k)$. So, this is a contradiction, and therefore, this assumption is wrong, and hence, every compact subset of $\mathbb{R}$ is bounded.

Moving ahead, let us see the proof of the well-known Heine-Borel theorem, which states that every closed and bounded subset of $\mathbb{R}$ is compact. The proof is simple. For it, let us take $A \subseteq \mathbb{R}$, given that it is closed and bounded. Now, if A is bounded, what can we conclude? We can conclude that this A will be contained in some closed interval $[a, b]$. Also, if $A \subseteq [a, b]$; again, just see this is given that this A is closed and we have already studied that $[a, b]$ is compact. So, what is here? This A will be compact. Why? We have already studied the fact that a closed or a closed subspace of a compact topological space is compact. Thus, every closed and bounded subset of $\mathbb{R}$ is compact.

Moving ahead, let us see the converse of the Heine-Borel theorem, which can also be proved easily. What is the converse of the Heine-Borel theorem? That is, every compact subset of $\mathbb{R}$ is closed and bounded. In order to prove it, let $A \subseteq \mathbb{R}$ be compact. We have to show that this is closed and bounded. One thing is to be noted here: $\mathbb{R}$ is with Euclidean topology. We have already studied that this space is Hausdorff. Note that if this is Hausdorff, what relationship do we have with us now? This $\mathbb{R}$ is Hausdorff, and this A is compact. So, finally, what do we have with us? We have a compact subspace of a Hausdorff space. Can we conclude from here that this A will always be closed? The answer is yes, because we have already studied that a compact subset of a Hausdorff space is closed. Therefore, A is closed. What we have to show now is that A is bounded. Note that we have already proved that every compact subset of $\mathbb{R}$ is bounded. Therefore, this A is bounded, too. Hence, every compact subset of $\mathbb{R}$ is closed as well as bounded.

Now, let us see the compactness of a subset $A \subseteq \mathbb{R}^n$. So, let us take $\mathbb{R}^n$ with standard topology, that is, the Euclidean topology, along with the standard metric, that is, Euclidean metric. Then, $A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded. In order to prove this theorem, we require two results: the first one is that we have already seen that the real line with Euclidean topology, this is Hausdorff, and it can be shown that $\mathbb{R}^n$ with the Euclidean topology, is also Hausdorff. Secondly, if we are having a metric space (X, d), let us take a subset $A \subseteq X$; the question is when we say that this A is bounded. The answer is, we say that A is bounded if there exists some real number $m > 0$ such that $d(x, y) \leq m$, for all $x, y \in A$. With these two results, let us try to prove this theorem.

Now, what we are going to do is first assume that A is compact, and now we have to show that this is closed and bounded. What we can do is just see. What this A is? A is a subset of $\mathbb{R}^n$. We know that $\mathbb{R}^n$ is Hausdorff and what A is? Note that A is compact. So, using the logic which we have used when we proved the converse of Heine Borel theorem, we can conclude that A is closed. Now, what do we have to show? We have to show that this A is bounded. In order to justify it, let us take an open cover $\mathcal{C} = \{B(0, n) : n \in \mathbb{N}\}$ of A. If this is an open cover of A, use the compactness of A. Because A is compact, we can conclude that $\mathcal{C}$ will have a finite subcover; that is, $A \subseteq B(0, n_1) \cup B(0, n_2) \cup ... \cup B(0, n_k)$. Now, let us see the nature of open balls here, these are centered at the origin with different radii, and that radius is a natural number. So, we can conclude from here that if we are taking a natural number m, which is the maximum of these $n_1, n_2, ..., n_k$, then $A \subseteq B(0, m)$, from here we can conclude that if we are taking any $x, y \in A$, what will be the $d(x, y) < 2m$. Thus, A is bounded.

Moving ahead, let us see the converse of this theorem. In order to prove the converse, what are we assuming? We are assuming that this A is closed as well as bounded, and our motive is to prove that it is compact. In order to prove it, let us take an element. So, what are we taking? We are taking an element in A. Let us take this $(a_1, a_2, ..., a_n) \in A$. It is to be noted that this A is bounded. So, if we are using the boundedness of A, what does it mean? It means that for all $x, y \in A$ there exists a real number $m > 0$ such that $d(x, y) < m$. Now, we are constructing a set P, which is the Cartesian product of closed intervals, that is $P = [a_1 - m, a_1 + m] \times [a_2 - m, a_2 + m] \times ... \times [a_n - m, a_n + m]$. Now, if we want to visualize the Cartesian product of two closed intervals in $\mathbb{R}^2$, let us see how it will look like. For example, this is our A, and we are taking an element (a_1, a_2) here. What we can do is that this will be included in this rectangle, that is, this is $a_1 - m$, and this is $a_1 + m$. Similarly, this is nothing but $a_2 - m$, this is $a_2 + m$. So, what will happen from here? We can conclude that A will always be contained in P, and if this $A \subseteq P$, what P is? P is nothing but the Cartesian product of closed intervals and what the closed intervals are? These are compact. So, what exactly is P? P is nothing but the product of compact spaces, and we know that the finite product of compact topological spaces is compact. Therefore, this P is compact. Now, what we have with us. We have this A, which is a subset of P. Note that this is given that A is closed, and we have shown that this P is compact. Now, we use the result, which is well known to us, that a closed subspace of a compact topological space is compact. Therefore, this A is compact. This is the proof of the converse part of this theorem. Hence, we have finally shown that a subset A of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

These are the references.

That's all from this lecture. Thank you.