Course Name: Essentials of Topology
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Welcome to Lecture 36 on Essentials of Topology.

In this lecture, we will continue the study of the concept of connectedness. Specifically, we will study some of the results associated with connectedness or disconnectedness of subsets in a topological space. Beginning with the concept of connected subsets. In the previous lecture, we have already seen that if we have a topological space $(X, \mathcal{T})$, and a subset $E \subseteq X$, E is connected if $(E, \mathcal{T}_E)$ is connected. Also, whenever we are talking about the disconnectedness of E, we say that E is disconnected if $(E, \mathcal{T}_E)$ is disconnected. We have also seen a number of examples.

Begin with a characterization of the disconnectedness of subsets; let us see a theorem, which is stated as: Let $(X, \mathcal{T})$ be a topological space and $E \subseteq X$. Then E is disconnected iff there exist $\mathcal{T}$-open sets A and B such that $E \subseteq A \cup B, E \cap A \neq \emptyset, E \cap B \neq \emptyset$, and $E \cap A \cap B = \emptyset$. In order to prove it, let us assume first that E is disconnected. Then, there exist $G, H \subseteq E$ such that $G \neq \emptyset, H \neq \emptyset$, G and H are $\mathcal{T}_E$-open, $G \cap H = \emptyset$, and $G \cup H = E$. Now, as G and H are $\mathcal{T}_E$-open; $G = E \cap A$, and $H = E \cap B$, where $A, B \in \mathcal{T}$. So, what have we shown? Now, we have with us two $\mathcal{T}$-open sets A and B. It is clear that $G \subseteq A$ and $H \subseteq B$. Now, $E = G \cup H \subseteq A \cup B$. Further, as G and H are nonempty, and $G = E \cap A$ and $H = E \cap B$, therefore $E \cap A \neq \emptyset$ and $E \cap B \neq \emptyset$. Finally, as $G \cap H = \emptyset$, $G = E \cap A$, and $H = E \cap B$. From here, we conclude that $E \cap A \cap B = \emptyset$.

Let us see the converse part. For which, assume that there exist $\mathcal{T}$-open sets A and B such that $E \subseteq A \cup B, E \cap A \neq \emptyset, E \cap B \neq \emptyset$, and $E \cap A \cap B = \emptyset$. Then, our motive is to prove that $(E, \mathcal{T}_E)$ is disconnected. Now, because we already have with us two $\mathcal{T}$-open sets, A and B; let us take $E \cap A = G$ and $E \cap B = H$. Obviously, G and H both are nonempty subsets of E, as $E \cap A \neq \emptyset$, and $E \cap B \neq \emptyset$. Further, G and H are $\mathcal{T}_E$-open. Also, $G \cap H = \emptyset$, because $G \cap H = (E \cap A) \cap (E \cap B) = E \cap A \cap B = \emptyset$. Finally,

$G\cup H = (E\cap A)\cup(E\cap B) = E\cap(A\cup B)$. But note that $E \subseteq A\cup B$. Therefore, $G \cup H = E$. So, what have we shown? We have justified that there exist two $\mathcal{T}_E$-open subsets of E, that create a separation of E. Therefore, $(E, \mathcal{T}_E)$ is disconnected. Let us take an example related to this concept.

We are taking the topology as Euclidean topology on the set of real numbers, and $E = \mathbb{Q}$. Also, let $A = (-\infty, \sqrt{2})$, and $B = (\sqrt{2}, \infty)$. It is clear that both A and B are $\mathcal{T}_e$-open. Also, $E \cap A = \mathbb{Q} \cap (-\infty, \sqrt{2}) \neq \emptyset$, and $E \cap B = \mathbb{Q} \cap (\sqrt{2}, \infty) \neq \emptyset$. It is to be noted here that $E \cap A \cap B = \emptyset$, because $A \cap B = \emptyset$. Also, $E \subseteq A \cup B$ as $\mathbb{Q} \subseteq (-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)$. Thus, all the criteria of the previous theorems are satisfied. Specifically, there exist two $\mathcal{T}_e$-open sets A and B such that $\mathbb{Q} \subseteq A \cup B$, $E \cap A \neq \emptyset$, $E \cap B \neq \emptyset$, and $E \cap A \cap B = \emptyset$. Therefore, $\mathbb{Q} \subseteq \mathbb{R}$, is disconnected. This is a justification regarding the disconnectedness of the set of rationals by using the previous theorem.

Moving ahead, let us discuss a simple result that is useful to characterize some theorems related to connectedness. The statement of this result is: Let E be a connected subset of a disconnected topological space $(X, \mathcal{T})$ having separation A and B. Then $E \subseteq A$ or $E \subseteq B$. Let us prove this theorem. As $(X, \mathcal{T})$ has a separation A and B, what does it mean? It means that A and B are subsets of X with the property that A is a nonempty set, B is a nonempty set, A and B are $\mathcal{T}_e$-open, $A \cap B = \emptyset$, and $A \cup B = X$. Now, we have a connected subset E of X. As $E \subseteq X$ and $X = A \cup B$, let us try to compute $(E \cap A) \cup (E \cap B)$. Note that $(E \cap A) \cup (E \cap B) = E \cap (A \cup B) = E \cap X = E$. Thus, $E = (E \cap A) \cup (E \cap B)$. It is to be noted here that $E \cap A, E \cap B \subseteq E$. Also, $E \cap A$ and $E \cap B$ are $\mathcal{T}_E$-open because A and B are $\mathcal{T}$-open. Finally, $(E \cap A) \cap (E \cap B) = E \cap (A \cap B) = \emptyset$. Thus, we have shown that there exist two subsets $E \cap A$ and $E \cap B$ of E such that E can be expressed as the union of these two subsets; these two are open and disjoint. Now, if $E \cap A \neq \emptyset$, and $E \cap B \neq \emptyset$, what can we conclude? We can say that E is disconnected. But note that we are given that E is a connected subset of X. So, if E is connected, it means that either $E \cap A = \emptyset$, or $E \cap B = \emptyset$. Now, if $E \cap A = \emptyset$, as $E = (E \cap A) \cup (E \cap B)$, we can conclude that $E = E \cap B$, or $E \subseteq B$. Similarly, if we are taking $E \cap B = \emptyset$, again $E = E \cap A$, or we can say that $E \subseteq A$.

Let us take an example to discuss this result. Let $X = \{a, b, c, d\}$ alongwith $\mathcal{T} = \{\emptyset, X, \{a, b\}, \{c, d\}\}$. It is clear that $\{a, b\}$ is an open subset of X. Also, the complement of open set $\{c, d\}$ is $\{a, b\}$, that is, $\{a, b\}$ is closed, too. This means that a proper subset of X exists, which is both closed and open. Therefore, $(X, \mathcal{T})$ is disconnected. Now, the separation of the set X is given as $\{a, b\}$ and $\{c, d\}$. As we already know that the singleton sets are always connected. So, the singleton set $\{a\}$ is connected, and note that $\{a\} \subseteq \{a, b\}$, i.e., $\{a\}$ is contained in one part of the separation of X. Similarly, if we are taking another one, let us take the singleton set $\{c\}$. Note that $\{c\} \subseteq \{c, d\}$. This is an example to provide details about the result. From this example, one more thing is clear. Let us take the set $E = \{a, b\}$, and check the connectedness of this set. Note that, $\mathcal{T}_E = \{\emptyset, E\}$. That is, $\{a, b\}$ is connected, too. Also, it is to be noted here that a superset of this $\{a, b\}$ is $\{a, b, c, d\}$. We have shown that $\{a, b\}$ is connected, but note that $\{a, b, c, d\}$ is disconnected. From here, we can conclude that a superset of a connected set may not be connected.

The question arises: can we impose some condition so that supersets of a connected set may be connected? The answer is given by this theorem, which is stated as: If E is a connected subset of a topological space $(X, \mathcal{T})$, and $F \subseteq X$ such that $E \subseteq F \subseteq \overline{E}$. Then F is connected. Meaning is to say that if E is connected, then all the subsets lying between E and $\overline{E}$ will always be connected. Let us justify it by contradiction. If possible, let F be disconnected. Then, as usual, we have already seen there exist two subsets, A and B of F, such that A and B are nonempty, A and B are $\mathcal{T}_F$-open, $A \cap B = \emptyset$, and $F = A \cup B$. Now, what is given to us is that E is connected, and here, $E \subseteq F$, which is disconnected. So, by using the previous lemma, we can deduce that $E \subseteq A$ or $E \subseteq B$. Now, if $E \subseteq A$, then $\overline{E} \subseteq \overline{A}$, or that $\overline{E} \cap B \subseteq \overline{A} \cap B = \emptyset$. It is to be noted that $\overline{A} \cap B = \emptyset$ because we have already studied a characterization of separated sets, that is given in terms of two disjoint open sets. If two sets A and B are disjoint and open, then $\overline{A} \cap B = \emptyset$. Also, $A \cap \overline{B} = \emptyset$. Therefore, $\overline{E} \cap B = \emptyset$. But it is to be noted here that $F \subseteq \overline{E}$. It means that $A \cup B \subseteq \overline{E}$, or $B \subseteq \overline{E}$. If $B \subseteq \overline{E}$, we can deduce from here that $\overline{E} \cap B = B$, this is a contradiction. Similarly, if $E \subseteq B$, we can again see that there will be a contradiction. Actually, we can conclude that $\overline{E} \subseteq \overline{B}$, or $\overline{E} \cap A \subseteq A \cap \overline{B}$, and because of the fact that $\overline{E} \cap A \subseteq A \cap \overline{B} = \emptyset$, that is, $\overline{E} \cap A = \emptyset$. Again, let us use the fact that $A \cup B \subseteq \overline{E}$. Therefore, $A \subseteq \overline{E}$, or we can say that the $\overline{E} \cap A = A$. So, in this case too, we are reaching a contradiction. Therefore, our assumption that F is disconnected is wrong; hence, F is connected. Now, if we are taking this $F = \overline{E}$, then $\overline{E}$ is also connected. In other words, the closure of a connected set is connected.

Let us try to justify this by an example. For example, if we are taking a set $X = \{a, b, c, d\}$ with the simplest topology, that is the indiscrete topology $\mathcal{T}$. We know that $(X, \mathcal{T})$ is connected. Also, if we are taking a set E as a singleton set $\{a\}$, we know that this E is connected. Now, let us see some of the supersets of $\{a\}$, as $\{a, b\}$, $\{a, b, c\}$, these are also connected. What about $\overline{\{a\}}$? Note that $\overline{\{a\}} = X$, which is connected. So, it justifies that if a set is connected, all the supersets of the set, which lie between the connected set and its closure, will be connected.

These are the references.

That's all from this lecture. Thank you.