Course Name: Essentials of Topology
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Welcome to Lecture 34 on Essentials of Topology.

In this lecture too, we will continue the study of homeomorphisms. From the definition of homeomorphism, we have already seen that how homeomorphism preserves the topological structures. In this lecture, we will study some of the properties of topological spaces which can be preserved under homeomorphisms. If these properties are preserved under homeomorphisms, such properties are known as topological properties. Begin with the concept of topological properties. Let us see what does it mean. A property of a topological space is called topological property if it is preserved under homeomorphism. Meaning is to say that if we are having a topological space $(X, \mathcal{T})$, which has some property P. Let us take any topological space $(Y, \mathcal{T}')$ so that there exists a homeomorphism between these two. If the topological space $(Y, \mathcal{T}')$ also has this property P, then we say that this P is a topological property.

Let us take some examples. Begin with the concept of metric spaces. We have already seen that the metric induces a topology known as metric topology, and such topological spaces are called metrizable. What we are going to discuss here is that metrizability is a topological property. In order to justify it, let us take a metric space (X, d), and now let us take the topology induced by this metric on X as $\mathcal{T}_d$. What we have to justify is that if we are taking any topological space $(Y, \mathcal{T}')$, which is homeomorphic to $(X, \mathcal{T}_d)$, that is, this $f : (X, \mathcal{T}_d) \to (Y, \mathcal{T}')$ is a homeomorphism, our motive is to show that $\mathcal{T}'$ can be induced by some metric. If this is so, we can say that metrizability is a topological property. In order to justify it, let us take a function ρ from $Y \times Y$ to the set of non-negative real numbers such that $\rho(y_1, y_2) = d(f^{-1}(y_1), f^{-1}(y_2))$, for all $y_1, y_2 \in Y$. Note that the function f is bijective. So, we can talk about $f^{-1}(y)$, for all $y \in Y$, and this $f^{-1}(y)$ is an element of X. We can justify that this ρ is a metric on Y, and if we want to justify it, let us see one by one. The first one is, as $\rho(y_1, y_2) = d(f^{-1}(y_1), f^{-1}(y_2))$, $\rho(y_1, y_2) > 0$ because it is given to us that d is a metric. Also, if this $\rho(y_1, y_2) = 0$, it means that $d(f^{-1}(y_1), f^{-1}(y_2)) = 0$. Because d is a metric, we conclude that $f^{-1}(y_1) = f^{-1}(y_2)$, and from here, we can conclude that $y_1 = y_2$, as f is an injective map, therefore f^{-1} too.

Moving ahead, this symmetric property also holds here because $\rho(y_1, y_2) = d(f^{-1}(y_1), f^{-1}(y_2)) = d(f^{-1}(y_2), f^{-1}(y_1))$, as d is metric. Therefore, it can be written as $\rho(y_2, y_1)$. Finally, coming to the additive property, $\rho(y_1, y_2)) = d(f^{-1}(y_1), f^{-1}(y_2)) \leq d(f^{-1}(y_1), f^{-1}(y_3)) + d(f^{-1}(y_3), f^{-1}(y_2))$, and this holds for all $y_3 \in Y$. Thus $\rho(y_1, y_2) \leq \rho(y_1, y_3) + \rho(y_3, y_2)$. Therefore, ρ is a metric on Y.

Moving ahead, as we have shown, this ρ is a metric on Y. Therefore, this will induce a topology on Y, and what is our motive? Our motive is to justify that this topology is nothing but $\mathcal{T}'$. Now, as ρ induces a topology on Y, let us take the basis. For this topology, it is given by $\mathcal{B}_\rho = \{B_\rho(y, r) : y \in Y, r > 0\}$. We have to justify two things. First, we have to justify that this $\mathcal{B}_\rho$ is a subfamily of this topology $\mathcal{T}'$, and after that, we have to justify that for all $G \in \mathcal{T}'$, and for all $y \in G$, there is $B_\rho(y, r) \in \mathcal{B}_\rho$ such that $B_\rho(y, r) \subseteq G$. Begin with first one. In order to justify it, we can show that if we are taking any $y \in Y$, $B_\rho(y, r) = f(B_d(f^{-1}(y), r))$. If we want to justify it, just take any $z \in B_\rho(y, r)$. Then, we can say that this $\rho(y, z) < r$, or from here, we can also say that $d(f^{-1}(y), f^{-1}(z)) < r$, or we can say that $f^{-1}(z) \in B_d(f^{-1}(y), r)$, or we can say that this $z \in f(B_d(f^{-1}(y), r))$. Thus, $B_\rho(y, r) \subseteq f(B_d(f^{-1}(y), r))$.

Again, if we are taking any $z \in f(B_d(f^{-1}(y), r))$, we can say that $f^{-1}(z) \in B_d(f^{-1}(y), r)$, and if this is so, $d(f^{-1}(y), f^{-1}(z)) < r$, or that $\rho(y, z) < r$, i.e., $z \in B_\rho(y, r)$. Thus, $f(B_d(f^{-1}(y), r)) \subseteq B_\rho(y, r)$. Now, as f is a homeomorphism, therefore f is open, and if f is open, $f(B_d(f^{-1}(y), r)) \in \mathcal{T}'$, or $B_\rho(y, r) \in \mathcal{T}'$. Thus, each element of this $\mathcal{B}_\rho$ is a member of $\mathcal{T}'$; this is the first step.

Moving ahead, now let us take any $G \in \mathcal{T}'$, and $y \in G$. Then, we can write that this $f^{-1}(y) \in f^{-1}(G)$. Again, by continuity of $f : (X, \mathcal{T}_d) \to (Y, \mathcal{T}')$, as f is a homeomorphism, we can conclude that $f^{-1}(G) \in \mathcal{T}_d$, that is $f^{-1}(G)$ is $\mathcal{T}_d$-open, and if this is $\mathcal{T}_d$-open, there exists $B_d(f^{-1}(y), r) \subseteq f^{-1}(G)$, or we can say that $f(B_d(f^{-1}(y), r)) \subseteq G$, or that $B_\rho(y, r) \subseteq G$. So, what we found? We have shown that for all $G \in \mathcal{T}'$, and for all $y \in G$, there exists

$B_\rho(y, r) \in \mathcal{B}_\rho$ such that $B_\rho(y, r) \subseteq G$. Thus, $\mathcal{B}_\rho$ is a basis for topology $\mathcal{T}'$. In another sense, we can say that this $\mathcal{T}'$ is a topology that is induced by metric ρ. Thus, we have shown that homomorphic image of a metrizable space is also metrizable. That's all about metrizability, which is a topological property.

Moving ahead, let us take another example of topological property. We have already studied the concept of first-countable spaces as well as second-countable spaces. We can show that these are topological properties. If you want to justify that these are topological properties, let us take only one, and the another can be proved similarly. So, we are going to justify only that second countability is a topological property. In order to justify it, let us take a topological space $(X, \mathcal{T})$, which is second countable. Now, let us take another topological space $(Y, \mathcal{T}')$. Also, let us take that there exists a homeomorphism $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$. Our motive is to justify that this $(Y, \mathcal{T}')$ is also second countable. Now, begin with the assumption. It is given to us that $(X, \mathcal{T})$ is second countable, and if this is second countable, there exists a countable basis for it. Let us take that countable basis to be $\mathcal{B}$. The question is how to construct a countable basis for $(Y, \mathcal{T}')$? The answer is that because we already have a countable structure associated with the topological space $(X, \mathcal{T})$, why not use this structure? So, what my motive is to say that take a collection $\mathcal{B}_1 = \{f(B) : B \in \mathcal{B}\}$. Note that $\mathcal{B}_1$ is countable. If we can justify that this is a basis for $(Y, \mathcal{T}')$, then we achieve what is our motive. In order to justify that this is a basis for $(Y, \mathcal{T}')$, one thing is clear: when we are taking $B \in \mathcal{B}$, $B \in \mathcal{T}$, and therefore $f(B) \in \mathcal{T}'$ because f is an open. So, we conclude that this $\mathcal{B}_1$ is always a sub-collection of $\mathcal{T}'$. Now, let us take $G \in \mathcal{T}'$ and $y \in G$. If $y \in G$, as the map is surjective; there exists $x \in X$ such that $f(x) = y$, or we can say that $f(x) \in G$. If $f(x) \in G$, $x \in f^{-1}(G)$. Again, by continuity of f, this $f^{-1}(G)$ is $\mathcal{T}$-open, and if this is $\mathcal{T}$-open, we can say that there exists $B \in \mathcal{B}$ such that $x \in B \subseteq f^{-1}(G)$, or $f(x) \in f(B) \subseteq G$, or $y \in f(B) \subseteq G$. Therefore, $\mathcal{B}_1$ is a basis for $(Y, \mathcal{T}')$. We have already discussed that $\mathcal{B}_1$ is countable. Therefore, $(Y, \mathcal{T}')$ is second countable, and second countability is a topological property. It is to be noted here that whenever we are using a homoemorphism, we have with us bijectiveness, openness, and continuity. Even, we are not using all the properties to justify that this particular property is topological property but actually our motive is here to justify that these properties are topological. Accordingly, we are always considering a homeomorphism.

Moving ahead, let us take one more example. This is separability, and separability is a topological property. If we want to justify that separability is a topological property, similar to the previous concepts, let us take a homeomorphism $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$. Also, we are taking this $(X, \mathcal{T})$ as a separable space. Our motive is to justify that $(Y, \mathcal{T}')$ is also separable. Now, as $(X, \mathcal{T})$ is separable, there exists $A \subseteq X$ so that the $\bar{A} = X$. Note that A is countable. As we have to justify that $(Y, \mathcal{T}')$ is separable, we have to find a countable subset of Y, whose closure is equal to Y. One thing is clear from here: because we already have a countable subset A of X, why not try to use it? So, it is natural that if we are taking $f(A)$, it is a subset of Y, and as A is countable, we can say that this $f(A)$ is countable. Can we justify that $\overline{f(A)} = Y$. If we can justify it, then $(Y, \mathcal{T}')$ is separable, and it can be shown in simple way because we already have with us that $\bar{A} = X$. Thus, $f(\bar{A}) = f(X)$. Note that f is surjective, so $f(X) = Y$, i.e., $f(\bar{A}) = Y$. We have shown that $f(\bar{A}) = \overline{f(A)}$ if and only if f is homomorphism. Therefore, we conclude that $\overline{f(A)} = Y$, and as $f(A)$ is a countable set, we have a countable dense subset of Y. Therefore, the topological space $(Y, \mathcal{T}')$ is separable.

Moving ahead, what have we seen till now? We have taken some of the examples of topological property. Let us take some of the examples which are not topological properties. So, the first such example is length. Length is not a topological property, but it is to be mentioned here that length is a geometric property. In order to justify it, let us take an example, and we have already seen that if we are taking a closed interval $[0, 1]$ and we are taking another closed interval, let us take $[2, 5]$; these are homeomorphic because we have already shown that two closed intervals are homeomorphic when we are discussing in terms of Euclidean topology. The length of this interval $[0, 1]$ is 1, and the length of this interval $[2, 5]$ is 3. Note that both are homeomorphic. That is, there exists a homeomorphism between $[0, 1]$ and $[2, 5]$, but their lengths are different. It means that if we are taking two objects of different lengths, they can be homeomorphic, and therefore length is not a topological property.

Moving ahead, we know the concept of boundedness. Note that this boundedness is not a topological property. We have already seen that the open interval, let us take this is $(0, 1)$, this is homeomorphic to the set of real numbers. The interval $(0, 1)$ is bounded, but $\mathbb{R}$ is unbounded. As there is a homeomorphism between these two, that is, open interval $(0, 1)$ and $\mathbb{R}$. Therefore, boundedness is not a topological property.

These are the references.

That's all from this lecture. Thank you.