Course Name: Essentials of Topology
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Week: 06
Lecture: 01

Welcome to Lecture 31 on Essentials of Topology.

In this lecture too, we will continue the study of continuous functions. In the previous lectures, we have seen the construction of some continuous functions, such as the constant function, inclusion function, and restricted function. In this lecture, we will also study the construction of some continuous functions. In the previous lecture, we have already seen that the projection maps π_1 and π_2 from product space, that is, $X_1 \times X_2$ with the product topology to topological spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are continuous.

A natural question is, if we are having a topological space, let us take $(A, \mathcal{T}')$, and we are having two topological spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$. If we are having a continuous function $f_1 : A \to X_1$ and another continuous function $f_2 : A \to X_2$, can we construct a continuous function from $(A, \mathcal{T}')$ to the product space? The answer is yes, and we can do it. Then, the question will be, how will this continuous function f depend on functions f_1 and f_2? The answer is given in the form of this theorem. The statement is: Let $f_1 : A \to X_1$ and $f_2 : A \to X_2$ be functions. Further, let $f : A \to X_1 \times X_2$ be a function such that $f(a) = (f_1(a), f_2(a)), a \in A$. Then f is continuous iff f_1 and f_2 are continuous.

Now, begin with let us take this A is equipped with a topology $\mathcal{T}'$, X_1 is with topology $\mathcal{T}_1$, and X_2 is with topology $\mathcal{T}_2$. Also, let us take the product topology on $X_1 \times X_2$ be denoted by $\mathcal{T}$. Now, assume that this f is continuous. Our motive is to prove that f_1 and f_2 are continuous. We already have a continuous function $f : A \to X_1 \times X_2$. Also, we have projection maps $\pi_1 : X_1 \times X_2 \to X_1$ and $\pi_2 : X_1 \times X_2 \to X_2$. Now, for $a \in A$, $(\pi_1 \circ f)(a) = \pi_1(f(a)) = \pi_1(f_1(a), f_2(a)) = f_1(a)$. Thus $\pi_1 \circ f = f_1$. Similarly, $\pi_2 \circ f = f_2$. Thus f_1 and f_2 are compositions of projection maps along with a continuous function f. But note that the projection maps are already continuous, and it is also given to us that f is continuous. Therefore, we can conclude that f_1 and f_2 are continuous functions.

Let us see the converse part of this result. Assume that f_1 and f_2 are continuous. Our motive is to prove that f is continuous. In order to prove that this f is continuous, let us use the characterization of continuity in terms of basis, that is if we are taking some basis element for the product topology and prove that its inverse image is open, that is, $\mathcal{T}'$-open, then this function f is a continuous function. For product topology, we already know that a basis element will be of the form $G_1 \times G_2$, where $G_1 \in \mathcal{T}_1$ and $G_2 \in \mathcal{T}_2$. Now, if we are computing $f^{-1}(G_1 \times G_2)$, what exactly will it be? We can see that this is precisely $f_1^{-1}(G_1) \cap f_2^{-1}(G_2)$. It is a set-theoretic concept that can be justified easily. Let us see it; for example, if $a \in f^{-1}(G_1 \times G_2)$, it means that $f(a) \in G_1 \times G_2$. But note that $f(a) = (f_1(a), f_2(a)) \in G_1 \times G_2$. From here, we can conclude that $f_1(a) \in G_1$ and this $f_2(a) \in G_2$, or from here, we can say that $a \in f_1^{-1}(G_1)$ and also $a \in f_2^{-1}(G_2)$, or $a \in f_1^{-1}(G_1) \cap f_2^{-1}(G_2)$. Thus, $f^{-1}(G_1 \times G_2) \subseteq f_1^{-1}(G_1) \cap f_2^{-1}(G_2)$. Similarly, $f_1^{-1}(G_1) \cap f_2^{-1}(G_2) \subseteq f^{-1}(G_1 \times G_2)$. Therefore, we can conclude that $f^{-1}(G_1 \times G_2) = f_1^{-1}(G_1) \cap f_2^{-1}(G_2)$. Now, this is already given to us that f_1 and f_2 are continuous functions. Note that what these functions are? f_1 is a function from $(A, \mathcal{T}')$ to $(X_1, \mathcal{T}_1)$, and what about f_2? f_2 is a function from $(A, \mathcal{T}')$ to $(X_2, \mathcal{T}_2)$. Now, if we are taking any $G_1 \in \mathcal{T}_1$, what about $f_1^{-1}(G_1)$? Note that $f_1^{-1}(G_1)$ is a member of $\mathcal{T}'$. Similarly, if we are taking any $G_2 \in \mathcal{T}_2$, what about this $f_2^{-1}(G_2)$, that is also a member of $\mathcal{T}'$. Thus, what we have? $f_1^{-1}(G_1) \in \mathcal{T}'$, and $f_2^{-1}(G_2) \in \mathcal{T}'$, where $G_1 \in \mathcal{T}_1$ and $G_2 \in \mathcal{T}_2$, or that, this $f_1^{-1}(G_1) \cap f_2^{-1}(G_2) \in \mathcal{T}'$, that is the inverse image of this $G_1 \times G_2$ under f is a $\mathcal{T}'$-open set. Hence, f is a continuous function.

Moving ahead, let us see the construction of some real-valued continuous functions. If $f, g : X \to \mathbb{R}$ are continuous functions. Then $f + g$, $f - g$, $f.g$ and $f/g, (g(x) \neq 0, \forall x \in X)$ are continuous. We know that the function $f + g$ is defined as $(f + g)(x) = f(x) + g(x)$. Also, $(f - g)(x) = f(x) - g(x)$. Similarly, $(f.g)(x) = f(x).g(x)$, and $(f/g)(x) = f(x)/g(x)$, provided $g(x)$ is not equal to zero, for all $x \in X$. If we want to justify the continuity of these functions, let us take, for example, the addition of the functions. We have two functions $f, g : X \to \mathbb{R}$ with us. Now, using our previous result, what can we conclude that we can construct a continuous function $h : X \to \mathbb{R} \times \mathbb{R}$ such that $h(x) = (f(x), g(x)), x \in X$. Now, we can define another function $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, which sends $(f(x), g(x))$ to $f(x) + g(x)$. Thus, what have we seen? We have seen that this $f + g$ is nothing but a composition of this addition function along with a new function that we have constructed, that is, h. Note from calculus that $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and by using our construction, that is, construction of h, h itself is a continuous function. Therefore, $f + g$ is continuous. Similarly, one can discuss the continuity of $f - g$, $f.g$ and f/g.

Moving to the next concept, this is known as Pasting Lemma, and provides an interesting way to construct a continuous function by using two given continuous functions with some conditions. The statement of this lemma is given here: Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be two topological spaces and $X = A \cup B$, where A and B are closed sets in X. Further, let $f : A \to Y$ and $g : B \to Y$ be continuous functions; and $f(x) = g(x)$, for all $x \in A \cap B$. Then the functions f and g combine to give a continuous function $h : X \to Y$ defined as under:

$$h(x) = \begin{cases} f(x), & \text{if } x \in A \\ g(x), & \text{if } x \in B. \end{cases}$$

Before proving this lemma, let us see an example. Let $f : (-\infty, 0] \to \mathbb{R}$ such that $f(x) = -x$, for all $x \in (-\infty, 0]$ and $g : [0, \infty) \to \mathbb{R}$ such that $g(x) = x$, for all $x \in [0, \infty)$. Now, let $A = (-\infty, 0]$ and $B = [0, \infty)$. It is to be noted here that we have assumed that $\mathbb{R}$ is equipped with Euclidean topology, and the topology on A is a relative topology with respect to Euclidean topology and the same concepts with B. Then the required conditions for Pasting Lemma are satisfied. Let us see one by one. It is clear from the definition of A as well as B, that $A \cup B = \mathbb{R}$, A and B are closed sets. Note that both f and g are continuous. Also, $A \cap B = \{0\}$, and $f(0) = g(0)$. Now, by Pasting lemma, we can construct a function $h : \mathbb{R} \to \mathbb{R}$ such that

$$h(x) = \begin{cases} -x, & \text{if } x \in (-\infty, 0] \\ x, & \text{if } x \in [0, \infty] \end{cases}$$

is continuous. In the assumptions, one thing is clear: first will always be required because, with the help of domains of f and g, we are defining a new function h, whose domain is X. Also, the continuity of f and g will be required to justify the continuity of h. The question is, what are the requirements for closedness of A and B, and equality of $f(x)$ and $g(x)$ at $A \cap B$?

Let us see through these examples. In the first example, we are taking a function $f : (-\infty, 0] \to \mathbb{R}$ such that $f(x) = x - 2$, for all $x \in (-\infty, 0]$ and $g : [0, \infty) \to \mathbb{R}$ such that $g(x) = x + 2$, for all $x \in [0, \infty)$. Note that here $A = (-\infty, 0]$, $B = [0, \infty)$, $A \cup B = \mathbb{R}$, there is no problem with the continuity, and even A and B are closed sets. But what is the problem? If we are looking for $A \cap B$, $A \cap B = \{0\}$. Note that $f(0) = -2$ and $g(0) = 2$. So, if we are defining $h : \mathbb{R} \to \mathbb{R}$ such that

$$h(x) = \begin{cases} x - 2, & \text{if } x \in (-\infty, 0] \\ x + 2, & \text{if } x \in [0, \infty) \end{cases}$$

by using the Pasting lemma, h is not a function. So, it justifies that we always require the condition that at the points of intersection, f as well as g, should be identical.

Moving ahead, if we make a minor change in the previous example. Let $f : (-\infty, 0) \to \mathbb{R}$ such that $f(x) = x - 2$, for all $x \in (-\infty, 0)$ and $g : [0, \infty) \to \mathbb{R}$ such that $g(x) = x + 2$, for all $x \in [0, \infty)$. Note that here $A = (-\infty, 0)$, $B = [0, \infty)$, $A \cup B = \mathbb{R}$, there is no problem with the continuity, and because $A \cap B$ is an empty set, trivially $f(x) = g(x)$, for all $x \in A \cap B$. Where is the problem? The problem is that this A is not a closed set. Still, by using the Pasting lemma, if we construct a function $h : \mathbb{R} \to \mathbb{R}$ such that

$$h(x) = \begin{cases} x - 2, & \text{if } x \in (-\infty, 0) \\ x + 2, & \text{if } x \in [0, \infty). \end{cases}$$

Then h is not a continuous function. Why? The answer is simple. If we are taking an open set, let us take $G = (1, 3)$, and if we are computing $h^{-1}(G)$. Then, $h^{-1}(G) = \{x \in \mathbb{R} : h(x) \in G\} = [0, 1)$. Note that $[0, 1)$ is not open in the Euclidean topology, therefore this h is not a continuous function.

Let us take one more example, where A and B both are not closed sets. Let $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x) = 1$, for all $x \in \mathbb{Q}$ and $g : \mathbb{Q}^c \to \mathbb{R}$ such that $g(x) = -1$, for all $x \in \mathbb{Q}^c$. Here, $A = \mathbb{Q}, B = \mathbb{Q}^c$. If we are using the Pasting lemma, then $h : \mathbb{R} \to \mathbb{R}$ such that

$$h(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ -1, & \text{if } x \in \mathbb{Q}^c. \end{cases}$$

Note that this h cannot be a continuous function, and here, the nature of A and B plays the key role because these are not closed sets. This h is not continuous because if we are taking $G = (0, 2)$, then $h^{-1}(G) = \mathbb{Q}$, which is not a member of the Euclidean topology, or this is not open, that's why this is not continuous. So, what we have seen through these examples is the importance of the assumptions that we have considered in the statement of the Pasting lemma.

Finally, let us see the proof of Pasting lemma. Note that the function are $f : (A, \mathcal{T}_A) \to (Y, \mathcal{T}')$, and $g : (B, \mathcal{T}_B) \to (Y, \mathcal{T}')$. What do we have to justify? We have to prove that $h : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous. As this is already given that A and B are closed sets, let us use the characterization of continuity in terms of closed sets. So, what exactly will we take? Let us take F as a $\mathcal{T}'$-closed set. If we are taking F as a $\mathcal{T}'$-closed set, our motive will be to show that $h^{-1}(F)$ is $\mathcal{T}$-closed. In order to justify that $h^{-1}(F)$ is $\mathcal{T}$-closed, let us see how to express this $h^{-1}(F)$ in terms of f as well as g. Note that this F is a subset of Y. By using simple set theory, we can justify that $h^{-1}(F) = f^{-1}(F) \cup g^{-1}(F)$. If we want to see it, let us take any $x \in h^{-1}(F)$. Then what will happen? We can write that $h(x) \in F$. Now, if $x \in A$, this $h(x) = f(x)$, and therefore $f(x) \in F$, or from here, $x \in f^{-1}(F)$. Similarly, if $x \in B$, $h(x) = g(x)$, and as $h(x) \in F$, $g(x) \in F$, or $x \in g^{-1}(F)$. Finally, we can conclude that $x \in f^{-1}(F) \cup g^{-1}(F)$. That is $h^{-1}(F) \subseteq f^{-1}(F) \cup g^{-1}(F)$. Now, if we are taking any $x \in f^{-1}(F) \cup g^{-1}(F)$, meaning is that either $f(x) \in F$ or this $g(x) \in F$, and from here we can conclude that $h(x) \in F$ or $x \in h^{-1}(F)$, that is this $f^{-1}(F) \cup g^{-1}(F) \subseteq h^{-1}(F)$. Thus, for all $F \subseteq Y$, $h^{-1}(F) = f^{-1}(F) \cup g^{-1}(F)$.

Now, we have to justify that $h^{-1}(F)$ is $\mathcal{T}$-closed and, in our hand, we already have $h^{-1}(F) = f^{-1}(F) \cup g^{-1}(F)$. Now, let us analyze the nature of $f^{-1}(F)$ as well as $g^{-1}(F)$. Note that F is $\mathcal{T}'$-closed. If F is $\mathcal{T}'$-closed, by continuity of f, $f^{-1}(F)$ is $\mathcal{T}_A$-closed, and if this is $\mathcal{T}_A$-closed, this $f^{-1}(F)$ can be written as $A \cap F'$, where F' is $\mathcal{T}$-closed. But it is to be noted here that this is already given that A is $\mathcal{T}$-closed. Thus $A \cap F'$ is $\mathcal{T}$-closed, or $f^{-1}(F)$ is $\mathcal{T}$-closed. Also, as g is a continuous function, we can justify that $g^{-1}(F)$ is $\mathcal{T}$-closed. Now, if $f^{-1}(F)$ and $g^{-1}(F)$ are $\mathcal{T}$-closed, we can conclude that $h^{-1}(F)$ is $\mathcal{T}$-closed, as it is a finite union of closed sets. So, what have we justified? We have shown that if we are taking a $\mathcal{T}'$-closed set, its inverse image under h is a closed set with respect to the topology $\mathcal{T}$. Therefore, the function h, which was defined in terms of f and g, is continuous; that's proof of Pasting lemma.

These are the references.

That's all from this lecture. Thank you.