Course Name: Essentials of Topology
Professor Name: S.P. Tiwari
Department Name: Mathematics & Computing
Institute Name: Indian Institute of Technology(ISM), Dhanbad
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Welcome to Lecture 21 on Essentials of Topology.

In this lecture, we will study the concept of product topology. This is another way to create a topology by using the given topologies. The actual idea is, if we are having two topological spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$. The question is: what topology can we put on $X_1 \times X_2$? The answer is: we can put a number of topologies, at least two are well known, that is, the discrete topology and indiscrete topology. However, our interest is to put a topology here by using these two topologies, $\mathcal{T}_1$ and $\mathcal{T}_2$.

One of the natural answers may be when we are talking about the Cartesian product of sets, why not let us take the collection $\{G_1 \times G_2 : G_1 \in \mathcal{T}_1, G_2 \in \mathcal{T}_2\}$. But the problem is, it may not be a topology because if we are taking two sets in this collection, let us take the first one as $G_1 \times G_2$ and the second as $H_1 \times H_2$. The problem is, if we are taking their union, that is, $(G_1 \times G_2) \cup (H_1 \times H_2)$, it may not belong to this collection because it may not be expressed in the form of $G \times H$, where $G \in \mathcal{T}_1$ and $H \in \mathcal{T}_2$. Let us try to understand this concept by using a diagram.

This one is for set X_1, and this is for set X_2. If we are taking $G_1 \times G_2$, let us take that this is a subset G_1 of X_1, and correspondingly this is a subset G_2 of X_2. Let us take another Cartesian product, that is, the Cartesian product of H_1 and H_2. So, this is a subset H_1 of X_1, and this is for a subset H_2 of X_2. From this diagram, it is clear that the union of $G_1 \times G_2$ and $H_1 \times H_2$ may not be expressed in the form of some $G \times H$, that is, the Cartesian product of two sets G and H, where G is from topology $\mathcal{T}_1$, and H is from the topology $\mathcal{T}_2$.

So, the question is how to construct the topology on $X_1 \times X_2$. Interestingly, this collection will form a basis for some topology on $X_1 \times X_2$. Let us see it. If we are having two topological spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$, take this collection, that is, $\mathcal{B} = \{G_1 \times G_2 : G_1 \in \mathcal{T}_1, G_2 \in \mathcal{T}_2\}$. We can show that this is a basis for a topology, and the topology we will obtain, that topology is known as the product topology on $X_1 \times X_2$. Let us see. It is clear from the definition that $X_1 \times X_2 \in \mathcal{B}$, as $X_1 \in \mathcal{T}_1$ and $X_2 \in \mathcal{T}_2$. Therefore, $X_1 \times X_2$ can be written as a union of some members of $\mathcal{B}$.

Coming to the second requirement. Let us take some $B_1 = G_1 \times G_2$ from this $\mathcal{B}$ and also take $B_2 = H_1 \times H_2$ from this $\mathcal{B}$. What we actually have to justify is that if we are taking this $B_1 \cap B_2$, that is, $(G_1 \times G_2) \cap (H_1 \times H_2)$, and let us take an element (x, y) of this $B_1 \cap B_2$, our motive is to search a $B_3 \in \mathcal{B}$ such that $(x, y) \in B_3 \subseteq B_1 \cap B_2$. The question is, how to find out this B_3? Now, as $(x, y) \in (G_1 \times G_2) \cap (H_1 \times H_2)$, from here we can conclude that $(x, y) \in G_1 \times G_2$ and this (x, y) is also a member of $H_1 \times H_2$, or we can say that $x \in G_1, y \in G_2$, and this $x \in H_1$ and $y \in H_2$. From here, we can also conclude that $x \in G_1 \cap H_1$ and $y \in G_2 \cap H_2$. What about this $G_1 \cap H_1$ and $G_2 \cap H_2$? Note that G_1 is from $\mathcal{T}_1$, G_2 is from $\mathcal{T}_2$. Also, H_1 is from $\mathcal{T}_1$, and H_2 is from $\mathcal{T}_2$. So, from here we can conclude that this $G_1 \cap H_1 \in \mathcal{T}_1$. Similarly, this $G_2 \cap H_2 \in \mathcal{T}_2$, and if this is the case, we can also write $(x, y) \in (G_1 \cap H_1) \times (G_2 \cap H_2)$. Let us take this as B_3. Now, this $B_3 \in \mathcal{B}$ as $G_1 \cap H_1 \in \mathcal{T}_1$ and $G_2 \cap H_2 \in \mathcal{T}_2$. Only one thing or only one step remains, that is, to justify that $B_3 \subseteq (G_1 \times G_2) \cap (H_1 \times H_2)$, which can be justified easily.

For example, if we are taking any $(p, q) \in B_3$ and that B_3 is nothing but $(G_1 \cap H_1) \times (G_2 \cap H_2)$. Then $p \in G_1 \cap H_1$, and $q \in G_2 \cap H_2$. From here, we can write that $p \in G_1$ and $p \in H_1$, and $q \in G_2$, and $q \in H_2$, or we can write that $(p, q) \in G_1 \times G_2$ and $(p, q) \in H_1 \times H_2$. If we are clubbing these two, we can conclude that this $(p, q) \in (G_1 \times G_2) \cap (H_1 \times H_2)$, and that is nothing but $B_1 \cap B_2$. So, we have proven both the criteria that should be satisfied for $\mathcal{B}$ to be a basis. Therefore, the collection $\mathcal{B}$ that we have taken forms a basis for some topology, and that topology is product topology. But if we look at the structure of the basis, we always look for a smaller structure than the topology, while the basis that we are constructing here is derived from topologies and has a bigger structure. The question arises: Can we reduce the size of this $\mathcal{B}$? The answer is yes and the answer is given by this result.

Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces with bases $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Then $\mathcal{B} = \{B_1 \times B_2 : B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}$ is a basis for product topology on $X_1 \times X_2$. Let us try to justify this result. In order to justify this result, let us take this as $\mathcal{T}$, the product topology on $X_1 \times X_2$. Now, if we are taking any $G \in \mathcal{T}$ and any member (x, y) of this G, what do we have to justify? We have to justify that there exists some $B \in \mathcal{B}$ such that $(x, y) \in B \subseteq G$. So, our motive is to justify this point. The question is, how do we justify it? Let us see.

If we are taking this $(x, y) \in G$, because we have already shown that there is a basis for the product topology, and that basis provides an element $G_1 \times G_2$ such that $(x, y) \in G_1 \times G_2 \subseteq G$. Note that this $G_1 \in \mathcal{T}_1$ and $G_2 \in \mathcal{T}_2$. From here, it is clear that $x \in G_1$, and obviously, this G_1 is from $\mathcal{T}_1$, and $y \in G_2$. Note that this $G_2 \in \mathcal{T}_2$. But we already have bases for $\mathcal{T}_1$ and $\mathcal{T}_2$. From there, we can conclude that there exists B_1 in the basis for topology $\mathcal{T}_1$ and a member B_2 from the basis for topology $\mathcal{T}_2$ such that $x \in B_1 \subseteq G_1$ and $y \in B_2 \subseteq G_2$, or we can say that $(x, y) \in B_1 \times B_2 \subseteq G_1 \times G_2$. Note that $G_1 \times G_2 \subseteq G$. So, what we have obtained, that is, $(x, y) \in B_1 \times B_2 \subseteq G$. Note that this $B_1 \in \mathcal{B}_1$, and this $B_2 \in \mathcal{B}_2$. Thus, this collection $\mathcal{B}$, which is given here, is a basis for product topology on $X_1 \times X_2$. Let us construct some examples.

The first and the simplest example is if we are taking two topological spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$. Let us take both the topologies as indiscrete. What about the product topology on $X_1 \times X_2$? Is it also indiscrete? The answer is yes. This will also be an indiscrete topology. Why? Because if we are looking at the basis structure for $\mathcal{T}_1$ and $\mathcal{T}_2$, the basis $\mathcal{B}$ for the product topology contains only $X_1 \times X_2$, and what is our X_1, X_1 is from $\mathcal{T}_1$ and X_2 is from $\mathcal{T}_2$. Thus the topology generated by this $\mathcal{B}$ is nothing but an indiscrete topology. Hence, we can conclude that the product of indiscrete spaces is indiscrete. Similar to this one, why not discuss about the discrete topology? Let us take two topological spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$, where $\mathcal{T}_1$ and $\mathcal{T}_2$ both are discrete. The question is whether this product topology is discrete. The answer is yes. This will also be a discrete topology. How? The basis for this topology $\mathcal{T}_1$ is $\mathcal{B}_1 = \{\{x_1\} : x_1 \in X_1\}$. Similarly, the basis for topology $\mathcal{T}_2$ is $\mathcal{B}_2 = \{\{x_2\} : x_2 \in X_2\}$. The question is, what about the basis for this product topology. From $\mathcal{B}_1$ and $\mathcal{B}_2$, we can construct a basis for this topology, that is, $\mathcal{B} = \{\{(x_1, x_2)\} : x_1 \in X_1, x_2 \in X_2\}$. As this is a collection of singleton subsets of $X_1 \times X_2$, the topology generated by this basis will be the discrete topology. Our final conclusion is, the product of two discrete topological spaces is discrete.

Let us take some more general examples. Beginning with the Euclidean topology on the set of reals. Let us take the product of Euclidean spaces. The question is, what about the basis for product topology on $\mathbb{R}^2$? The answer is, if we are looking or if we are recalling the basis structure for Euclidean topology, we have seen that the collection of open intervals forms a basis for Euclidean topology. So, if it is an interval (a, b), and similarly if here is in the interval (c, d). The basis for product topology on $\mathbb{R}^2$, that will be given by a collection of open rectangles. Moving to another example, let us try to see the basis for product topology on $\mathbb{R}^2$, when the topologies are lower limit topology on $\mathbb{R}$. So, the basis element can be visualized by using this figure. Because we know that the basis elements for this lower limit topology look like in this way, these are semi-open intervals. We are taking semi-open intervals for this lower limit topology, that is $[a, b)$ and $[c, d)$. So, how this basis element will look like? This will look like in this way. This will be closed from the lower side this will also be closed from this left side, but this will be open from the upper side and open from right side. So, finally, this is a basis element for the product topology on $\mathbb{R}^2$, where $\mathbb{R}$ is endowed with the lower limit topologies. So, finally this is an image of a basis element for the product topology on $\mathbb{R}^2$, when lower limit topologies are with the set of real numbers.

Moving ahead, we have already seen the basis. Actually, we have seen two bases for product topology. Let us try to reduce the size further by using the concept of sub-basis. Actually, this concept is based on the notion of two maps π_1 as well as π_2. Let $\pi_1 : X_1 \times X_2 \to X_1$ and $\pi_2 : X_1 \times X_2 \to X_2$ be maps such that $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. These maps are called projection maps. From the definition, we have some observations. The first one is, both the maps, that is π_1 and π_2 are subjective maps. Also, if we are taking a subset G_1 of X_1 and we compute this inverse image of G_1 under π_1, that is a collection $\{(x_1, x_2) : \pi_1(x_1, x_2) \in G_1\}$, or this can be written as $\{(x_1, x_2) : x_1 \in G_1\}$, and from here we can write that this set is nothing but $G_1 \times X_2$. Similarly, if we are taking a subset G_2 of X_2 and we are computing $\pi_2^{-1}(G_2)$, that can be written as $X_1 \times G_2$.

With these concepts, let us move ahead. Let us see one more interesting thing about the intersection of this $\pi_1^{-1}(G_1)$ and $\pi_2^{-1}(G_2)$. Now, $\pi_1^{-1}(G_1)$, which we have already seen that this is $G_1 \times X_2$. Note that G_1 is a subset of X_1 and $\pi_2^{-1}(G_2)$, that is given as $X_1 \times G_2$, where G_2 is a subset of X_2. So, let us take this is for set X_1 and this is for set X_2. Also, this region is representing set $G_1 \times X_2$. If this is representing $G_1 \times X_2$, actually this is our set G_1. Similarly, let us take this region, and this region is representing $X_1 \times G_2$. So, actually this is a subset G_2 of X_2. From this diagram, it is clear that the intersection of these two is this particular region, and this region is nothing but $G_1 \times G_2$.

Now, having this idea in mind, let us see how a set $\mathcal{B}'$ becomes a sub-basis for product topology. Actually, the set is given by $\mathcal{B}' = \{\pi_1^{-1}(G_1) : G_1 \in \mathcal{T}_1\} \cup \{\pi_2^{-1}(G_2) : G_2 \in \mathcal{T}_2\}$. We can justify that this $\mathcal{B}'$ is a sub-basis for the product topology on $X_1 \times X_2$. Let us denote by $\mathcal{T}$ the product topology on $X_1 \times X_2$. Also, let us denote by $\mathcal{T}'$ the topology generated by this $\mathcal{B}'$. Our motive is to justify that these $\mathcal{T}$ and $\mathcal{T}'$, both will be the same. So, what we have to justify is, justify that this $\mathcal{T}'$ is coarser than $\mathcal{T}$ and also justify that $\mathcal{T}$ is coarser than $\mathcal{T}'$. If we are taking any $G \in \mathcal{T}'$, what about this G? G will be nothing but the union of a finite intersection of members of $\mathcal{B}'$. Note that how the members of $\mathcal{B}'$ will look like. They will be of the form, that is $G_1 \times X_2$ and $X_1 \times G_2$. We have already seen that these are also members of this basis for product topology. Therefore, these will also belong to $\mathcal{T}$, and if these are in $\mathcal{T}$, by the feature of or by the definition of topology, the union of their finite intersection will be a member of $\mathcal{T}$. Therefore, this $G \in \mathcal{T}$, and it justifies that this $\mathcal{T}'$ is coarser than $\mathcal{T}$.

If we want to justify that $\mathcal{T}$ is coarser than $\mathcal{T}'$, as we know that the sets of the form $G_1 \times G_2$ are members of the basis for the product topology. Note that we have seen on the previous slide that this $G_1 \times G_2$ is nothing, but this is the intersection of $\pi_1^{-1}(G_1) \cap \pi_2^{-1}(G_2)$ and note that these $\pi_1^{-1}(G_1)$ and $\pi_2^{-1}(G_2)$, these are members from $\mathcal{B}'$. Therefore, these will be the members of the basis for this topology, that is, $\mathcal{T}'$, and thus, this $G_1 \times G_2$ will be a member of the basis for the topology $\mathcal{T}'$. Hence, this topology $\mathcal{T}$ will be coarser than $\mathcal{T}'$, and hence, this $\mathcal{B}'$ is a sub-basis for the product topology.

Moving ahead, we have seen two concepts for the creation of a new topology: one is the concept of products, and the second is the concept of subspaces. Let us try to think about some relationship between these two concepts. The relationship is stated as under: Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces, with subspaces $(A_1, \mathcal{T}_{1A_1})$ and $(A_2, \mathcal{T}_{2A_2})$, respectively. Then the product topology on $A_1 \times A_2$ is the same as the subspace topology on $A_1 \times A_2$ as a subset of $X_1 \times X_2$.

Let us try to justify. Note that when we are talking about this A_1 with the relative topology on it, this relative topology will be given as a collection $\{A_1 \cap G_1 : G_1 \in \mathcal{T}_1\}$. Similarly, whenever we are looking for this subspace, that is $(A_2, \mathcal{T}_{2A_2})$, here this topology is given as a collection $\{A_2 \cap G_2 : G_2 \in \mathcal{T}_2\}$. Now, if we are trying to find out the product topology on $A_1 \times A_2$, the basis for this topology, let us take this is some $\mathcal{B}$ that will be given by the cartesian product of members of these topologies, that is $\{(A_1 \cap G_1) \times (A_2 \cap G_2) : G_1 \in \mathcal{T}_1, G_2 \in \mathcal{T}_2\}$.

Now, coming to the next when we are considering this $A_1 \times A_2$ as a subset of $X_1 \times X_2$. Note that $X_1 \times X_2$ is endowed with the product topology. So, the basis for relative topology on $A_1 \times A_2$, let us take that this is $\mathcal{B}^*$, and this is given by $\{(A_1 \times A_2) \cap (G_1 \times G_2) : G_1 \in \mathcal{T}_1, G_2 \in \mathcal{T}_2\}$. So, what have we got? We have two topologies on $A_1 \times A_2$ with their bases. One basis is given by $\mathcal{B}$, and another basis is given by B^*. But interestingly, both will be equal, that is what we can justify that $(A_1 \cap G_1) \times (A_2 \cap G_2)$ is the same as $(A_1 \times A_2) \cap (G_1 \times G_2)$.

If we want to justify it, let us take some element $(p, q) \in (A_1 \cap G_1) \times (A_2 \cap G_2)$. So, from here, we can conclude that $p \in A_1 \cap G_1$ and $q \in A_2 \cap G_2$, or $p \in A_1$ and $p \in G_1$, also $q \in A_2$ and $q \in G_2$. From here, we can write that $(p, q) \in A_1 \times A_2$ and this $(p, q) \in G_1 \times G_2$, or we can conclude that this $(p, q) \in (A_1 \times A_2) \cap (G_1 \times G_2)$. So, $(A_1 \cap G_1) \times (A_2 \cap G_2) \subseteq (A_1 \times A_2) \cap (G_1 \times G_2)$, and similarly, we can go in the reverse direction, and we can justify that $(A_1 \times A_2) \cap (G_1 \times G_2) \subseteq (A_1 \cap G_1) \times (A_2 \cap G_2)$. It means that this $\mathcal{B}$ and $\mathcal{B}^*$ both are the same, and that's the justification for this result that the product topology on $A_1 \times A_2$ is the same as the subspace topology on $A_1 \times A_2$, when $A_1 \times A_2$ is considered as a subset of $X_1 \times X_2$.

These are the references.

That's all from this lecture. Thank you.