Course Name: Essentials of Topology
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Welcome to Lecture 11 on Essentials of Topology.

In this lecture, we will study the concept of comparison of topologies. In the previous lectures, we have seen that there exists a number of topologies on a given set X. The question is, can we establish some relationship among such topologies? One of the answers is given in terms of comparison of topologies.

Formally, for two topologies, $\mathcal{T}_1$, and $\mathcal{T}_2$, on a non-empty set X, we say that $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$, or $\mathcal{T}_2$ is finer than $\mathcal{T}_1$ if each element of $\mathcal{T}_1$ is in $\mathcal{T}_2$, i.e., each $\mathcal{T}_1$-open set is a $\mathcal{T}_2$-open set, or that, for all $G \in \mathcal{T}_1$, $G \in \mathcal{T}_2$. In case $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$, but equality doesn't hold, we say that $\mathcal{T}_1$ is strictly coarser than $\mathcal{T}_2$, or we can also say that $\mathcal{T}_2$ is strictly finer than $\mathcal{T}_1$.

Begin with two examples. We are familiar with the indiscrete and discrete topologies on a set X. As we know, for a given set X, indiscrete topology is a collection of empty set and X, and it is the smallest topology on X. So, we can say that this topology will always be coarser than all the topologies on X. Also, coming to discrete topology, we know that for a given set X, the discrete topology is nothing but the power set of X, and this is the largest topology. Being the largest topology on a set X, this will obviously be finer than all the topologies on a given set X.

Moving ahead, let us take one more example. Assume $X = \{a, b, c\}$ and

- $\mathcal{T}_1 = \{\emptyset, X, \{a\}\}$,
- $\mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}$,
- $\mathcal{T}_3 = \{\emptyset, X, \{a, b\}, \{a\}, \{a, c\}\}$,

If we are trying to compare the topologies $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$. Then it is clear from the first and second that $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$, as the empty set, X, and singleton set $\{a\}$, are all present in $\mathcal{T}_2$. Similarly, if we want to compare $\mathcal{T}_1$ and $\mathcal{T}_3$, we can see that $\mathcal{T}_1$ is coarser than $\mathcal{T}_3$, because the empty set, X, and singleton set $\{a\}$ are also in $\mathcal{T}_3$. But the question is, what about $\mathcal{T}_2$ and $\mathcal{T}_3$? The problem is that $\{b, c\}$ is in $\mathcal{T}_2$, which is not in $\mathcal{T}_3$, and similarly, $\{a, c\}$ is in $\mathcal{T}_3$, which is not in $\mathcal{T}_2$. Meaning is, $\mathcal{T}_2$ and $\mathcal{T}_3$ are not comparable. Thus, whenever we have a number of topologies on a set, it may happen that one topology is coarser/finer than others, and it may also be possible that topologies cannot be compared.

Let us take some topologies on the set of real numbers. The first one is Euclidean topology, which we have already seen. This topology is strictly coarser than the lower limit topology and the upper limit topology. The question is how? It can be justified like, let us take any $G \in \mathcal{T}_e$. Then, we have seen that for all $x \in G$, there exist two real numbers, a and b, with $a < b$, such that $x \in (a, b) \subseteq G$. Now, if we want to visualize it, this structure will look like: if this is the real line and this is a subset $G \subseteq \mathbb{R}$. If we are taking $x \in G$, what can we do? We can find a and b such that $x \in (a, b) \subseteq G$. From here, we can construct two things. The first one is that $x \in [x, b) \subseteq G$. If this is happening, we can conclude that G is a member of the lower limit topology. Also, we can write that $x \in (a, x] \subseteq G$ and from here, we can conclude that G is in the upper limit topology. Therefore, this Euclidean topology is coarser than the lower limit and the upper limit topologies.

Note that this relationship will be strict. Because we have already seen that if we are taking the semi-open interval $[2, 3)$, this is a member of the lower limit topology, but this interval $[2, 3)$ is not in the Euclidean topology. Similarly, the interval $(2, 3]$, is in the upper limit topology, but the same interval cannot be a member of Euclidean topology. Therefore, this relation is true.

Moving ahead, there is no comparison between the lower limit topology and the upper limit topology. This is because the interval $[2, 3)$ is in lower limit topology, but note that this interval cannot be a member of the upper limit topology, which we have already discussed. Similarly, if we are taking the interval $(2, 3]$, this is in the upper limit topology, but the same interval is not a member of the lower limit topology. Therefore, we conclude that these two topologies are not comparable.

Further, on the set of real numbers, again, let us discuss about cofinite topology, Euclidean topology, and cocountable topology. Interestingly, cofinite topology is strictly coarser than both topologies. Let us see it. For example, if we are taking any G in this cofinite topology. Then as we know that G^c is finite, i.e., if we are taking this G^c as a set $\{x_1, x_2, ..., x_n\}$, $x_1, x_2, ..., x_n \in \mathbb{R}$, and $x_1 < x_2 < ... < x_n$. Now, if we are talking about G, which is nothing but $(-\infty, x_1) \cup (x_1, x_2) \cup ...(x_{n-1}, x_n) \cup (x_n, \infty)$. We have already seen that such intervals are in Euclidean topology. Therefore, we can say that G is a member of $\mathcal{T}_e$, as it is the union of open intervals. But note that we can find some members in Euclidean topology, for example, this open interval $(2, 3)$. This open interval is not a member of cofinite topology, because its complement is not finite.

Moving to the next one, let us discuss cofinite topology and co-countable topology on $\mathbb{R}$. If we are taking any G in the cofinite topology, then G^c is finite, or we can also conclude that G^c is countable. Thus, we can say that this G is a member of co-countable topology. But note that $\mathbb{Q}^c$ is in co-countable topology, as its complement will be nothing but $\mathbb{Q}$ itself. Therefore, $\mathbb{Q}^c$ is in co-countable topology. However, $\mathbb{Q}^c$ is not a member of the cofinite topology because the complement of it will be $\mathbb{Q}$, and this is not finite. Therefore, $\mathcal{T}_{cc}$, that is, co-countable topology is strictly finer than the cofinite topology $\mathcal{T}_{cf}$.

Moving ahead, if we are looking for co-countable topology and Euclidean topology, both are not comparable, why? The answer is here. If we are taking this $\mathbb{Q}^c$, this is in co-countable topology because its complement is countable. But note that we have already seen that this is not a member of Euclidean topology because for $x \in \mathbb{Q}^c$ we cannot construct open interval (a, b) such that $x \in (a, b) \subseteq \mathbb{Q}^c$. Similarly, if we are taking this open interval $(2, 3)$ in the Euclidean topology. Note that this open interval is not a member of the co-countable topology because its complement is not countable. Therefore, these topologies are not comparable. Even co-countable topology and lower limit topology are not comparable. Also, co-countable topology and upper limit topology are not comparable. The justification is similar to what we have seen in the case of co-countable topology and Euclidean topology.

Let us summarize what we have seen till now. So, this D denotes the discrete topology on the set of reals, and I denotes the indiscrete topology on the set of real numbers. We have seen that the cofinite topology is coarser than the Euclidean topology. Also, we already know that the indiscrete topology is coarser than all the topologies. So, we can say that the Euclidean topology and the cofinite topology are finer than the indiscrete topology.

Now, coming to the next one, we have also seen co-countable topology, lower limit topology, and upper limit topology. We have seen that the cofinite topology is coarser than co-countable topology, Euclidean topology is coarser than the lower limit topology, and even Euclidean topology is coarser than the upper limit topology. Also, co-countable topology, lower limit topology, and upper limit topology are not comparable. So, there is no relationship among these three. Finally, as discrete topology is finer than all the topologies, we can say that co-countable topology, lower limit topology, and upper limit topology are coarser than the discrete topology.

When we have studied this comparison, that is, when one topology is contained in another, a natural question arises: Can we talk about the union and intersection of two topologies? Yes, we can talk. Then, the question will be whether the union of topologies will also be a topology or the intersection of topologies will also be a topology. In the first case, the answer is negative; that is, the union of two topologies may not be a topology.

Let us see it through an example. For example, X is a set; this is containing three elements a, b, and c. Let us take the first topology, $\mathcal{T}_1$, as a collection of the empty set, X, and singleton set $\{b\}$. Let us take another topology, $\mathcal{T}_2$, as a collection of the empty set, X and $\{c\}$. If we find their union, that is, $\mathcal{T}_1 \cup \mathcal{T}_2$, it will be a collection of the empty set, X, $\{b\}$ and $\{c\}$. But note that this one is not a topology on this X, because $\{b\}$ and $\{c\}$ are members of this $\mathcal{T}_1 \cup \mathcal{T}_2$. The question is, what about $\{b\} \cup \{c\} = \{b, c\}$. This is not a member of $\mathcal{T}_1 \cup \mathcal{T}_2$. Therefore, the union of two topologies may not be a topology.

Moving to the next one, the intersection of two topologies on a set is a topology. Here is no problem. Let us see it. For example, we are having a non-empty set X with two topologies, $\mathcal{T}_1$ and $\mathcal{T}_2$. Our motive is to justify that this intersection is also a topology on X. Note that the empty set will always be a member of this intersection because the empty set is a member of $\mathcal{T}_1$, and the empty set is also a member of $\mathcal{T}_2$. Similarly, X is a member of this intersection as X belongs to $\mathcal{T}_1$ and X also belongs to $\mathcal{T}_2$.

Moving to the next, if we are taking a finite number of members $G_1, G_2, ..., G_n$ of $\mathcal{T}_1 \cap \mathcal{T}_2$. Then what will happen? These $G_1, G_2, ..., G_n$ are members of $\mathcal{T}_1$, $G_1, G_2, ..., G_n$ are also members of $\mathcal{T}_2$. But note that both are topologies. So, we can conclude that their intersection is in $\mathcal{T}_1$, and $G_1 \cap G_2 \cap ... \cap G_n$ is also in $\mathcal{T}_2$. From these two, we can conclude that $G_1 \cap G_2 \cap ... \cap G_n \in \mathcal{T}_1 \cap \mathcal{T}_2$.

Finally, let us take an indexed family of sets $\{G_i : i \in I\}$ indexed by I, where every G_i is in $\mathcal{T}_1 \cap \mathcal{T}_2$. Then, our motive is to justify that $\cup\{G_i : i \in I\}$, is also a member of $\mathcal{T}_1 \cap \mathcal{T}_2$. This is similar to what we have seen in the case of intersections, and it can be seen in a simple way. Because every $G_i \in \mathcal{T}_1 \cap \mathcal{T}_2$, meaning is, $G_i \in \mathcal{T}_1$ and $G_i \in \mathcal{T}_2$, for all $i \in I$. So, from here, we can conclude that $\cup\{G_i : i \in I\}$, is in $\mathcal{T}_1$, and $\cup\{G_i : i \in I\}$ is also in $\mathcal{T}_2$ because $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies. From here, we can say that the $\cup\{G_i : i \in I\}$, is also a member of $\mathcal{T}_1 \cap \mathcal{T}_2$. So, we have seen all the requirements for $\mathcal{T}_1 \cap \mathcal{T}_2$ to be a topology on X. Therefore, the intersection of two topologies on a set is also a topology. Even this result can be generalized, which states that for some index set I, if we are taking $i \in I$, and let $\mathcal{T}_i$ be a topology on X. Then, the intersection of all such topologies is also a topology on X. That is, an arbitrary intersection of topologies is also a topology on the given set.

These are the references.

That's all from today's lecture. Thank you.