Advanced Engineering Mathematics
Lecture 9

1 Extremum of two-variable function and Euler's theorem application

Example 1.1. Find the maxima and minima of $f(x) = x^3 - 4x^2$.

Sol. Given

$$f(x) = x^3 - 4x^2$$
$$\Rightarrow f'(x) = 3x^2 - 8x$$

For extrema/ extremum,

$$f'(x) = 0 \Rightarrow 3x^2 - 8x = 0 \Rightarrow x = 0, x = \frac{8}{3}.$$

Again,

$$f''(x) = 6x - 8$$
$$f''(0) = -8 < 0$$
$$f''(\frac{8}{3}) = 6 \times \frac{8}{3} - 8 = 16 - 8 = 8 > 0.$$

At, $x = 0, f''(x) < 0$ which implies that the function attains its maximum value at $x = 0$.
At, $x = \frac{8}{3}, f''(x) > 0$, which implies that the function attains its minimum value at $x = \frac{8}{3}$.
So, the maximum value of

$$f = f(x)\Big|_{x=0}$$
$$= 0$$

And the minimum value of

$$f = f(x)\Big|_{x=\frac{8}{3}}$$
$$= x^3 - 4x^2\Big|_{x=\frac{8}{3}}$$
$$= \frac{8^3}{27} - \frac{4.8^2}{9}$$
$$= \frac{8^2}{9} \times \left[\frac{8}{3} - 4\right]$$
$$= \frac{8^2}{9} \times \left(-\frac{4}{3}\right)$$
$$= -\frac{256}{27}$$

Example 1.2. Find the maximum and minimum values of the function,

$$f(x, y) = x^3 - y^3 + 3y - 3x$$

Sol. Here,

$$f_x(x, y) = 3x^2 - 3$$
$$f_y(x, y) = -3y^2 + 3$$

Suppose the extremum values are attained at the point (x_0, y_0). Then for the extremum,

$$f_x(x_0, y_0) = 0 \Rightarrow 3x_0^2 - 3 = 0 \Rightarrow x_0^2 = 1 \Rightarrow x_0 = \pm 1$$
$$f_y(x_0, y_0) = 0 \Rightarrow -3y_0^2 + 3 = 0 \Rightarrow y_0^2 = 1 \Rightarrow y_0 = \pm 1$$

The extremum points are $P_1(1,1)$, $P_2(1,-1)$, $P_3(-1,1)$, $P_4(-1,-1)$. For maximum and minimum values, we need the 2nd derivatives and the discriminant D,

$$f_{xx} = 6x, f_{yy} = -6y, f(x,y) = 0, f_{yx} = 0$$
$$D = AB - C^2 = 6x \times (-6y) - 0 = -36xy$$

Now,

$$D\Big|_{P_1(1,1)} = -36 \times 1 \times 1 = -36 < 0, A = 6, B = -6, C = 0 \Rightarrow \text{ Saddle Point}$$
$$D\Big|_{P_2(1,-1)} = 36 > 0, A = 6, B = 6, C = 0 \Rightarrow \text{ Minimum Value}$$
$$D\Big|_{P_3(-1,1)} = 36 > 0, A = -6, B = -6, C = 0 \Rightarrow \text{ Maximum Value}$$
$$D\Big|_{P_4(-1,-1)} = -36 < 0, A = -6, B = 6, C = 0 \Rightarrow \text{ Saddle Point}$$
$$f(x,y)\Big|_{(1,-1)} = 8, f(x,y)\Big|_{(-1,1)} = 4$$

Example 1.3. For the function

$$f(x,y) = x^3 - y^3 + 3y - 3x$$

Find out the Taylor's series expansion about the point $P(2,1)$.

Sol.

$$f(x,y) = x^3 - y^3 + 3y - 3x \Rightarrow f(x,y)\Big|_{P(2,1)} = 4$$
$$f_x(x,y) = 3x^2 - 3 \Rightarrow f_x(x,y)\Big|_{P(2,1)} = 9$$
$$f_y(x,y) = -3y^2 + 3 \Rightarrow f_y(x,y)\Big|_{P(2,1)} = 0$$
$$f_{xx}(x,y) = 6x \Rightarrow f_{xx}(x,y)\Big|_{P(2,1)} = 12$$
$$f_{yy}(x,y) = -6y \Rightarrow f_{yy}(x,y)\Big|_{P(2,1)} = -6$$
$$f_{xy}(x,y) = 0 \Rightarrow f_{xy}(x,y)\Big|_{P(2,1)} = 0$$

By Taylor's series,

$$f(x,y) = f((2,1)) + \frac{\partial f}{\partial x}\Big|_{(2,1)}(x-2) + \frac{\partial f}{\partial y}\Big|_{(2,1)}(y-1) + \frac{1}{2}\Big(\frac{\partial^2 z}{\partial x^2}\Big|_{(2,1)}(x-2)^2 + \frac{\partial^2 z}{\partial y^2}\Big|_{(2,1)}(y-1)^2$$
$$+ 2\frac{\partial^2 z}{\partial x \partial y}\Big|_{(2,1)}(x-2)(y-1)\Big) + ...$$
$$= 4 + 9 \times (x-2) + 0 \times (y-1) + 12 \times \frac{(x-2)^2}{2} + (-6) \times \frac{(y-1)^2}{2} + 0 \times (x-2)(y-1) + ...$$
$$= 4 + 9(x-2) + 6(x-2)^2 - 3(y-1)^2 + ...$$

Example 1.4. If $u = \tan^{-1} \frac{x^3+y^3}{x-y}$, prove that $x^2 u_{xx} + 2xy u_{xy} + y^2 = (1 - 4\sin^2 u)\sin 2u$

Sol. Let, $v = \frac{x^3+y^3}{x-y} = \tan u$. Here, v is a homogeneous function of x and y of degree 2. Then by Euler's theorem,

$$\begin{aligned}
& x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 2v = 2\tan u \\
\Rightarrow\ & x\frac{\partial v}{\partial u}\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial u}\frac{\partial u}{\partial y} = 2v = 2\tan u \\
\Rightarrow\ & \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\sec^2 u = 2\tan u \\
\Rightarrow\ & x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u
\end{aligned}$$

Differentiating with respect to x and y respectively,

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = 2\cos 2u \frac{\partial u}{\partial x} \tag{1.1}$$

$$x\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = 2\cos 2u \frac{\partial u}{\partial y} \tag{1.2}$$

Now (1.1) $\times\, x +$ (1.2) $\times\, y$ we get,

$$\begin{aligned}
x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} + \sin 2u &= 2\cos 2u \sin 2u \\
&= \sin 2u(2\cos 2u - 1) \\
&= (1 - 4\sin^2 u)\sin 2u
\end{aligned}$$