Advanced Engineering Mathematics
Lecture 8

1 Functions of two or more than one variable

Let $z = f(x, y)$ then,

$$z_x = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}, \; z_y = \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y},$$
$$z_{xx} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial x}) = \frac{\partial^2 z}{\partial x^2}, z_{yy} = \frac{\partial}{\partial y}(\frac{\partial z}{\partial y}) = \frac{\partial^2 z}{\partial y^2}, z_{yx} = \frac{\partial}{\partial y}(\frac{\partial z}{\partial x}) = \frac{\partial^2 z}{\partial y \partial x} \cdots,$$
$$\frac{\partial}{\partial y}(\frac{\partial^{n-1}}{\partial y^{n-1}} z) = \frac{\partial^n z}{\partial y^n}, \frac{\partial}{\partial x}(\frac{\partial^{n-1}}{\partial x^{n-1}} z) = \frac{\partial^n z}{\partial x^n}.$$

Example 1.1. $z = f(x, y) = x^3 + 3xy^4 - 3x^4y^3$. Find z_x, z_y, z_{xx}, z_{yy}, z_{yx}, z_{xy}.

Sol.

$$\begin{aligned} z_x &= 3x^2 + 3y^4 - 12x^3y^3, \\ z_y &= 12xy^3 - 9x^4y^2, \\ z_{xx} &= 6x - 36x^2y^3, \\ z_{yy} &= 36xy^2 - 24x^4y, \\ z_{yx} &= 12y^3 - 36x^3y^2, \\ z_{xy} &= 12y^3 - 36x^3y^2 \end{aligned}$$

Taylor's Series: The Taylor's series of a function $f(x, y)$ about a point (x_0, y_0) provide an approximation of a function in the nbd of (x_0, y_0) and

$$z(x,y) = z(x_0,y_0) + \frac{\partial z}{\partial x}\Big|_{(x_0,y_0)}(x-x_0) + \frac{\partial z}{\partial y}\Big|_{(x_0,y_0)}(y-y_0) + \frac{1}{2}\Big(\frac{\partial^2 z}{\partial x^2}\Big|_{(x_0,y_0)}(x-x_0)^2 + \frac{\partial^2 z}{\partial y^2}\Big|_{(x_0,y_0)}(y-y_0)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\Big|_{(x_0,y_0)}(x-x_0)(y-y_0)\Big) + \ldots$$

Example 1.2. Let us take the function, $z(x, y) = x^3 + 3y - y^3 - 3x$. Find z_x, z_y, z_{xx}, z_{yy}, z_{yx}, z_{xy}.
Sol. $z_x = 3x^2 - 3$, $z_y = 3 - 3y^2$, $z_{xx} = 6x$, $z_{yy} = -6y$, $z_{yx} = 0$, $z_{xy} = 0$.

2 Maxima and Minima

Stationary Points: Maxima and Minima:
If $f'(x) = 0$ at $x = x_0$ and $f''(x)\Big|_{x=x_0} > 0$ then f attains minimum value at $x = x_0$. Similarly, if $f'(x) = 0$ at $x = x_0$ and $f''(x)\Big|_{x=x_0} < 0$ then f attains maximum value at $x = x_0$.

If f is a function of two variables, $\frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial y} = 0 \Rightarrow x = x_0, y = y_0$. Then we will calculate second order derivatives of it at point (x_0, y_0).

Let, $z_{xx}\Big|_{(x_0,y_0)} = A$, $z_{yy}\Big|_{(x_0,y_0)} = B$, $z_{xy}\Big|_{(x_0,y_0)} = C$. Compute the discriminant: $D = BC - A^2$

Case I. If $D > 0$ and $\frac{\partial^2 z}{\partial x^2}|_{(x_0,y_0)} > 0$ or $\frac{\partial^2 z}{\partial y^2}|_{(x_0,y_0)} > 0$, then z attains local minima at point (x_0, y_0).

Case II. If $D > 0$ and $\frac{\partial^2 z}{\partial x^2}|_{(x_0,y_0)} < 0$ or $\frac{\partial^2 z}{\partial y^2}|_{(x_0,y_0)} < 0$, then z attains local maxima at point (x_0, y_0).

Case III. If $D < 0$, then (x_0, y_0) is a saddle point.