Advanced Engineering Mathematics
Lecture 7

1 Functions of two or more than one variable

Example 1.1. Let, $f(x,y) = \begin{cases} x\sin\frac{1}{y} + \frac{x^2-y^2}{x^2+y^2} & y \neq 0 \\ 0 & y = 0 \end{cases}$

Show that $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$ exists but neither of $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ nor $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$ exists.

Sol. We shall show that $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist. Since $(x,y)\to(0,0)$, let, $y = mx$ be the path.

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2-m^2x^2}{x^2+m^2x^2} = \lim_{x\to 0} \frac{1-m^2}{1+m^2} = \frac{1=m^2}{1+m^2} \quad (x \neq 0)$$

So, the limit is path dependant, i.e., the limit is different for different values of m.
Again, $\lim\limits_{y\to 0} \sin\frac{1}{y}$ does not exist, then $\lim\limits_{x\to 0}\lim\limits_{y\to 0} x\sin\frac{1}{y}$ does not exist therefore, $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$ does not exist.

Now, $\lim\limits_{x\to 0} x\sin\frac{1}{y} = 0$ and $\lim\limits_{x\to a} f(x,y) = -1$, then $\lim\limits_{y\to 0}(0-1) = -1$

i.e., $\lim\limits_{y\to 0}\lim\limits_{x\to 0}(x\sin\frac{1}{y} + \frac{x^2-y^2}{x^2+y^2}) = -1$

Partial Derivatives for functions of two or more variables. $D \subset \mathbb{R}$ is usual domain of f and let $P(a,b) \in D$ be arbitrary. Let N be an nbd and $Q(a+h, b+k) \in N$. Then if α be an angle that the line joining $\vec{PQ}$ makes with positive $x-$axis, then the direction cosines of $\vec{PQ}$ are $l = \cos\alpha, m = \sin\alpha$. If $\rho = \sqrt{h^2+k^2}$. Then $h = \rho\cos\alpha, k = \rho\sin\alpha$ and $\rho \to 0$ as $Q \to P$. Now if,

$$\lim_{\rho\to 0} \frac{f(a+p\cos\alpha, b+p\sin\alpha) - f(a,b)}{\rho} \quad \text{exists.}$$

Then, it is called derivative of $f(x,y)$ at (a,b) in direction of α and it is denoted by $D_\alpha f(a,b)$. If $\alpha = 0$, then derivative is denoted by $\frac{\partial f}{\partial x}$ and if $\alpha = \frac{\pi}{2}$, then the derivative is denoted by $\frac{\partial f}{\partial y}$. These derivatives are called partial derivatives.
Continuity of $f(x,y)$. Let $N \subset D \subset \mathbb{R}^2$, where D is the domain of and N is the nbd of a point $(a,b) \in D$. Then f is said to be continuous at (a,b) if $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. Equivalently, $\forall\varepsilon > 0 \ \exists\ \delta > 0$ such that $|f(x,y) - f(a,b)| < \varepsilon$ whenever $|x-a| < \delta$ and $|y-b| < \delta$ or $(x-a)^2 + (y-b)^2 < \delta^2$
or,
$\lim\limits_{(h,k)\to(0,0)} f(a+h, b+k) = f(a,b)$ where $(a+h, b+k) \in D$.

Example 1.2. Verify the continuity of $f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ at $(0,0)$.

Sol. Here,

$$|f(x,y) - f(0,0)| = |xy\frac{x^2-y^2}{x^2+y^2} - 0| = |xy\frac{x^2-y^2}{x^2+y^2}| = |x||y||\frac{x^2-y^2}{x^2+y^2}| \leq |x||y| \leq \frac{x^2+y^2}{2} < \varepsilon,$$

if $x^2+y^2 < \delta^2 (= 2\varepsilon) \Rightarrow f$ is continuous at $(0,0)$.

Example 1.3. Verify the continuity of $f(x,y) = \begin{cases} x\sin\frac{1}{y} + y\sin\frac{1}{x} & xy \neq 0 \\ 0 & xy = 0 \end{cases}$ at $(0,0)$.

Sol. Here,

$$|f(x,y) - f(0,0)| = |x\sin\frac{1}{y} + y\sin\frac{1}{x} - 0| \leq |x||\sin\frac{1}{y} + |y|\sin\frac{1}{x} \leq |x| + |y| < \varepsilon$$

if $|x - 0| < \delta$, and $|y - 0| < \delta (= \frac{\varepsilon}{2}) \Rightarrow f$is continuous at $(0,0)$.

Example 1.4. Verify the continuity of $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & x^2 + y^2 \neq 0 \\ 0 & x^2 + y^2 = 0 \end{cases}$ at $(0,0)$.

Partial Derivatives (Computations). If $\lim_{h\to 0} \dfrac{f(x+h,y) - f(x,y)}{h}$ exists, then it is denoted by $\frac{\partial f}{\partial x}$ or f_x similarly, if $\lim_{k\to 0} \dfrac{f(x,y+k) - f(x,y)}{k}$ exists, then it is denoted by $\frac{\partial f}{\partial y}$ or f_y.

Example 1.5. Let, $f(x,y) = x^2y + 3xy$. Find f_x, f_y.

Sol. $f_x = 2xy + 3y$ and $f_y = x^2 + 3x$.