Adavnced Engineering Mathematics
Lecture 59

Example. Let the given curve $C : |z| = 3$, then evaluate the integral $\displaystyle\int_C \frac{e^{2z}}{(z+1)^4}\, dz$.

Solution: The given contour is the circle $x^2 + y^2 = 9$. Here $f(z) = \frac{e^{2z}}{(z+1)^4}$. Clearly, $z = -1$ lies within C. The function is not analytic at $z = -1$. Let $F(z) = e^{2z}$ and $n = 3$. Then by Cauchy integral formula of higher order

$$F^3(-1) = \frac{3!}{2\pi i} \int_C \frac{e^{2z}}{(z+1)^4}\, dz.$$

$$F(z) = e^{2z}, \quad F'(z) = 2e^{2z}, \quad F''(z) = 4e^{2z}, \quad F'''(z) = 8e^{2z}.$$

Therefore, from the higher order Cauchy integral formula, we have

$$\int_C \frac{e^{2z}}{(z+1)^4}\, dz = \frac{2\pi i}{6} \cdot 8e^{-2} = \frac{8\pi i}{3} e^{-2}.$$

Example. Let $C : |z| = 2$, then evaluate the integral $\displaystyle\int_C \frac{z}{(9-z^2)(z+i)}\, dz$.

Solution: The given contour is the circle $x^2 + y^2 = 4$. Here $f(z) = \frac{z}{9-z^2}$. At $z = -i$, the integrand is undefined/does not exist. We take $a = -i$, then by Cauchy integral formula

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz$$

$$\int_C \frac{z}{(9-z^2)(z+i)}\, dz = 2\pi i f(a) = \frac{\pi}{5}.$$

Example. Evaluate the integral $\displaystyle\int_C \frac{dz}{z(z+\pi i)}$, where C is the contour given by $|z + 3i| = 1$.

Solution: The given contour is $|z + 3i| = 1$, i.e., the circle $x^2 + (y+3)^2 = 1$. Let $f(z) = \frac{1}{z}$, and $a = -\pi i$. Then

$$\int_C \frac{f(z)\, dz}{(z + i\pi)} = 2\pi i f(-\pi i) = -2.$$

Example. Evaluate $I = \displaystyle\int_C z\, dz$, where C is the curve along the line $z = 0$ to $z = 1 + i$.

Solution:

$$I = \int_{z=0}^{1+i} z\, dz = \frac{(1+i)^2}{2}.$$

Example. Evaluate $\displaystyle\int_C (x - y + ix^2)\, dz$, where C is along the line $z = 0$ to $z = 1 + i$.

Solution: Consider the path $y = x$, then $dy = dx$. Therefore, $dz = (1 + i)\, dx$. Hence

$$I = \int_0^1 (x - x + ix)(1+i)\, dx = i(1+i) \int_0^1 x^2\, dx = \frac{i(1+i)}{3}.$$