Advanced Engineering Mathematics
Lecture 52

Example. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where $\vec{F}(x,y,z) = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$, and S is the part of surface of the sphere $x^2+y^2+z^2=1$ which lies in the first octant.

Solution: A vector normal to the surface $S : x^2+y^2+z^2 = 1$ is given by $\vec{\nabla}$, where $\phi(x,y,z) = x^2+y^2+z^2-1$. The normal vector to the surface $\vec{n} = \vec{\nabla}\phi = 2(x\,\hat{i} + y\,\hat{j} + z\,\hat{k})$.

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2(x\,\hat{i} + y\,\hat{j} + z\,\hat{k})}{2\sqrt{x^2+y^2+z^2}} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}.$$

Taking the projection of S into the xy-plane, then

$$\begin{aligned}
\iint_S \vec{F}\cdot\hat{n}\,ds &= \iint_R (yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k})\cdot(x\,\hat{i} + y\,\hat{j} + z\,\hat{k})\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}\\
&= \iint_R 3xyz\,\frac{dx\,dy}{z} = \iint_R 3xyz\,dx\,dy\\
&= \int_{r=0}^{1}\int_{\theta=0}^{\frac{\pi}{2}} r\cos\theta\, r\sin\theta\, r\,dr\,d\theta\\
&= \frac{3}{2}\int_{r=0}^{1} r^3\,dr\int_{\theta=0}^{\frac{\pi}{2}}\sin 2\theta\,d\theta\\
&= \frac{3}{2}\Big[\frac{r^4}{4}\Big]_0^1\Big[-\frac{\cos 2\theta}{2}\Big]_0^{\frac{\pi}{2}} = \frac{3}{8}.
\end{aligned}$$

Example. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where $\vec{F}(x,y,z) = z\,\hat{i} + x\,\hat{j} - 3y^2z\,\hat{k}$, and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant $z=0$ and $z=5$.

Solution: Let us take $\phi(x,y,z) = x^2+y^2-16$, then $\vec{\nabla}\phi = 2(x\,\hat{i} + y\,\hat{j})$ and $\hat{n} = \frac{2(x\,\hat{i}+y\,\hat{j})}{2\sqrt{x^2+y^2}} = \frac{x\,\hat{i}+y\,\hat{j}}{4}$.

We take R as the projection of S in xz-plane. Then

$$\begin{aligned}
\iint_S \vec{F}\cdot\hat{n}\,ds &= \iint_R (z\,\hat{i} + x\,\hat{j} - 3y^2z\,\hat{k})\cdot\frac{x\,\hat{i}+y\,\hat{j}}{4}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{j}|}\\
&= \iint_R \frac{(xz+xy)}{4}\,\frac{4\,dx\,dy}{y} \qquad \text{where } y = \sqrt{16-x^2}\\
&= \int_{z=0}^{5}\int_{x=0}^{4}\frac{xz + x\sqrt{16-x^2}}{\sqrt{16-x^2}}\,dx\,dy\\
&= \int_{z=0}^{5}\Big[-z\sqrt{16-x^2} + \frac{x^2}{2}\Big]_0^4 dz\\
&= \int_{z=0}^{5}(4z+8)\,dz = 90.
\end{aligned}$$

Example. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where $\vec{F}(x,y,z) = y\,\hat{i} + 2x\,\hat{j} - z\,\hat{k}$, and S is the surface of the plane $x+y=6$ in the first octant cut by the plane $z=4$.

Solution: Let us take $\phi(x,y,z) = 2x + y - 6$, then $\vec{n} = \vec{\nabla}\phi = (2\hat{i} + \hat{j})$ and $\hat{n} = \frac{1}{\sqrt{5}}(2\hat{i} + \hat{j})$. Taking the projection on xz-plane,

$$\begin{aligned}
I_S = \iint_S \vec{F} \cdot \hat{n}\, ds &= \iint_R \frac{1}{\sqrt{5}}(2y + 2x)\, \frac{dx\, dz}{|\hat{n} \cdot \hat{j}|} \\
&= \int_{z=0}^{4} \int_{x=0}^{3} \big[2x + 2(6 - 2x)\big]\, dx\, dz \\
&= 2 \int_{z=0}^{4} \int_{x=0}^{3} (6 - x)\, dx\, dz \\
&= 2 \cdot 4 \cdot \big[18 - \frac{9}{2}\big] = 108.
\end{aligned}$$