Advanced Engineering Mathematics
Lecture 51

Line integral: Any integral which is evaluated along a curve is called a line integral.

Let $\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}$ be the given equation of the curve C, joining the two points A at $t = t_1$ and B at $t = t_2$. Suppose $\vec{F}(x, y, z) = F_x\,\hat{\imath} + F_y\,\hat{\jmath} + F_z\,\hat{k}$ is a vector function and continuous along C. If s denotes the arc length of C, then $\frac{d\vec{r}}{ds} = \hat{t}$, unit tangent vector to C at the point P. Then the component of $\vec{F}$ along the tangent is $\vec{F} \cdot \frac{d\vec{r}}{ds}$. The integral of $\vec{F} \cdot \frac{d\vec{r}}{ds}$ along C from A to B is given by

$$\int_A^B \big[\vec{F} \cdot \frac{d\vec{r}}{ds}\big]\, ds = \int_A^B \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt = \int_{t_1}^{t_2} \big[F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt}\big]\, dt.$$

Circulation: If C is a simple closed curve (i.e., a curve which does not interest itself anywhere), then the tangent line integral of $\vec{F}$ around C is called the circulation of $\vec{F}$ about C. It is often denoted by

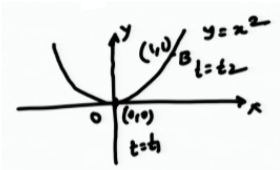
$$\Gamma = \oint_C \vec{F} \cdot d\vec{r} = \oint_C (F_x\, dx + F_y\, dy + F_z\, dz).$$

Example. Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y) = x^2\,\hat{\imath} + y^3\,\hat{\jmath}$ and C is the arc of the curve $y = x^2$ in xy-plane from $(0, 0)$ to $(1, 1)$.

Solution: C is the curve $y = x^2$ joining $(0, 0)$ to $(1, 1)$. The parametric form of the curve is given by

$$x = t, \quad y = t^2, \qquad \text{where } 0 \le t \le 1.$$

The line integral



$$\begin{aligned} I = \int_C \vec{F} \cdot d\vec{r} &= \int_C (x^2\,\hat{\imath} + y^3\,\hat{\jmath}) \cdot (dx\,\hat{\imath} + dy\,\hat{\jmath}) \\ &= \int_0^1 (t^2\,\hat{\imath} + t^6\,\hat{\jmath}) \cdot (\hat{\imath} + 2t\,\hat{\jmath})\, dt \\ &= \int_0^1 (t^2 + 2t^7)\, dt \\ &= \Big[\frac{t^3}{3} + \frac{2t^8}{8}\Big]_0^1 = \frac{7}{12}. \end{aligned}$$

Example. If $\vec{F}(x, y) = 3xy\,\hat{\imath} - y^2\,\hat{\jmath}$, then evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is the curve in the xy-plane $x^2 + y^2 = 25$ from $(5, 0)$ to $(0, 5)$.

Solution: C is the line segment of the circle $x^2 + y^2 = 25$ joining $(5, 0)$ to $(0, 5)$. The parametric form of the curve is given by

$$x = 5\cos t, \quad y = 5\sin t, \qquad \text{where } 0 \le t \le \frac{\pi}{2}.$$

The line integral

$$
\begin{aligned}
I = \int_C \vec{F} \cdot d\vec{r} &= \int_C (3xy\,\hat{i} - y^2\,\hat{j}) \cdot (dx\,\hat{i} + dy\,\hat{j}) \\
&= \int_0^{\frac{\pi}{2}} (3 \times 25 \cos t \sin t\,\hat{i} - 25 \sin^2 t\,\hat{j}) \cdot (-\sin t\hat{i} + \cos t\,\hat{j})\, dt \\
&= \int_0^{\frac{\pi}{2}} (-75 \sin^2 t \cos t - 25 \sin^2 t \cos t)\, dt \\
&= \int_0^{\frac{\pi}{2}} -100 \sin^2 t\, d(\sin t) \\
&= \Big[-\frac{100 \sin^3 t}{3}\Big]_0^{\frac{\pi}{2}} = -\frac{100}{3}.
\end{aligned}
$$

Example. Evaluate $\int_C (x\,dy - y\,dx)$ around the circle $x^2 + y^2 = 1$.

Solution: C is the circle $x^2 + y^2 = 1$. The parametric form of the curve is given by

$$x = \cos t, \quad y = \sin t, \qquad \text{where } 0 \leq t \leq 2\pi.$$

Therefore, the line integral

$$
\begin{aligned}
I &= \int_C (x\,dy - y\,dx) \\
&= \int_0^{2\pi} (\cos t \times \cos t + \sin t \times \sin t)\, dt \\
&= \int_0^{2\pi} dt = 2\pi.
\end{aligned}
$$

Example. Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y, z) = xy\,\hat{i} + yz\,\hat{j} + zx\,\hat{k}$ and the given curve $\vec{r} = t\,\hat{i} + t^2\,\hat{j} + t^3\,\hat{k}$, where t runs from -1 to 1.

Solution: C is the curve $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ where the parametric form of the curve is given by

$$x = t, \; y = t^2, \; z = t^3, \qquad \text{where } -1 \leq t \leq 1.$$

Therefore, the line integral

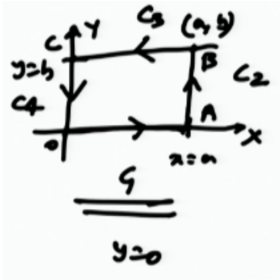
$$
\begin{aligned}
I = \int_C \vec{F} \cdot d\vec{r} &= \int_C (xy\,\hat{i} + yz\,\hat{j} + zx\,\hat{k}) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}) \\
&= \int_{-1}^{1} (t^3\,\hat{i} + t^5\,\hat{j} + t^4\,\hat{k}) \cdot (\hat{i} + 2t\,\hat{j} + 3t^2\,\hat{k})\, dt \\
&= \int_{-1}^{1} (t^3 + 5t^6)\, dt \\
&= \Big[\frac{t^4}{4} + \frac{5t^7}{7}\Big]_{-1}^{1} = \frac{10}{7}.
\end{aligned}
$$

Example. Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y) = (x^2 + y^2)\,\hat{i} - 2xy\,\hat{j}$, where C is the rectangle in the xy-plane $y = 0$, $x = 0$, $y = b$, and $x = a$.

Solution:

$$\int_C \vec{F} \cdot d\vec{r} = \underbrace{\int_{C_1} \vec{F} \cdot d\vec{r}}_{I_1} + \underbrace{\int_{C_2} \vec{F} \cdot d\vec{r}}_{I_2} + \underbrace{\int_{C_3} \vec{F} \cdot d\vec{r}}_{I_3} + \underbrace{\int_{C_4} \vec{F} \cdot d\vec{r}}_{I_4}$$

Along the curve C_1: $y = 0$, and x varies from 0 to a. Therefore,



$$I_1 = \int_0^a (x^2\,\hat{\imath} + 0\hat{\jmath}) \cdot (dx\,\hat{\imath} + 0\hat{\jmath})$$
$$= \int_0^a x^2\, dx = \frac{a^3}{3}.$$

Along the curve C_2: $x = a$, and y varies from 0 to b. Therefore,

$$I_2 = \int_0^b \left((a^2 + y^2)\,\hat{\imath} - 2ay\,\hat{\jmath}\right) \cdot (0\hat{\imath} + dy\,\hat{\jmath})$$
$$= \int_0^b -2ay\, dy = -ab^2.$$

Along the curve C_3: $y = b$, and x varies from a to 0. Therefore,

$$I_3 = \int_a^0 \left((x^2 + b^2)\,\hat{\imath} - 2bx\,\hat{\jmath}\right) \cdot (dx\,\hat{\imath} + 0\,\hat{\jmath})$$
$$= \int_a^0 (x^2 + b^2)\, dx = -\frac{a^3}{3} - ab^2.$$

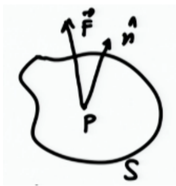
Along the curve C_4: $x = 0$, and y varies from b to 0. Therefore,

$$I_3 = \int_b^0 \left(y^2\,\hat{\imath} + 0\,\hat{\jmath}\right) \cdot (0\,\hat{\imath} + dy\,\hat{\jmath})$$
$$=0.$$

Therefore, $\int_C \vec{F} \cdot d\vec{r} = -2ab^2$.

Surface integral: Any integral which is evaluated over a surface S is called the surface integral.

$$I_S = \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_S \vec{F} \cdot d\vec{s}.$$



To evaluate the integral, first we take the projection,

$$I_S = \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_S \vec{F} \cdot \hat{n}\, \frac{dx\, dy}{|\hat{n} \cdot \hat{k}|} \qquad \text{on } xy\text{-plane}$$
$$= \iint_S \vec{F} \cdot \hat{n}\, \frac{dy\, dz}{|\hat{n} \cdot \hat{\imath}|} \qquad \text{on } yz\text{-plane}$$
$$= \iint_S \vec{F} \cdot \hat{n}\, \frac{dx\, dz}{|\hat{n} \cdot \hat{\jmath}|} \qquad \text{on } xz\text{-plane}$$