Advanced Engineering Mathematics
Lecture 5

1 Taylor's theorem

Theorem 1.1. *(Taylor's Theorem with General Form of Remainder) If a real valued function defined on* $[a, b]$ *or* $[a, a+h]$ *where* $a + h = b$ *be such that*
i) f^{n-1} *is continuous on* $[a, a+h]$*,*
ii) f^n *exists on* $(a, a+h)$*,*
then there exists a positive proper fraction $\theta(0 < \theta < 1)$ *such that*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(n) + R_n,$$

where $R_n = \frac{h^n(1-\theta)^{n-p}}{(n-1)!p} f^n(a+\theta h)$*,* p *being a positive integer* $\leq n$*.*

Theorem 1.2. *(Taylor's Theorem with Cauchy Form of Remainder) If a real valued function defined on* $[a, b]$ *or* $[a, a+h]$ *where* $a + h = b$ *be such that*
i) f^{n-1} *is continuous on* $[a, a+h]$*,*
ii) f^n *exists on* $(a, a+h)$*,*
then there exists a positive proper fraction $\theta(0 < \theta < 1)$ *such that*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(n) + R_n,$$

where $R_n = \frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)$ *is called the Cauchy's form of Remainder.*

Theorem 1.3. *(Taylor's Theorem with Lagranges Form of Remainder) If a real valued function defined on* $[a, b]$ *or* $[a, a+h]$ *where* $a + h = b$ *be such that*
i) f^{n-1} *is continuous on* $[a, a+h]$*,*
ii) f^n *exists on* $(a, a+h)$*,*
then, there exists a positive proper fraction $\theta(0 < \theta < 1)$ *such that*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(n) + R_n,$$

where $R_n = \frac{h^n}{n!} f^n(a+\theta h)$ *is called the Lagranges form of Remainder.*

Theorem 1.4. *(Maclaurin's Theorem) Let a function* $f : [0, x] \to \mathbb{R}$ *be such that*
i) f^{n-1} *is continuous on* $[0, x]$*,*
ii) f^n *exists on* $(0, x)$*,*
then there exists a positive proper fraction $\theta(0 < \theta < 1)$ *such that*

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{h^n(1-\theta)^{n-p}}{(n-1)!p}f^n(\theta h),$$

where p *is a positive integer* $\leq n$*.*

Hence, $f(x) = f(0) + \sum_{r=1}^{n-1} \frac{x^r}{r!} f^r(0) + \frac{h^n(1-\theta)^{n-p}}{(n-1)!p} f^n(\theta h)$.

Example 1.1. Expand the function $f(x) = e^x$ about the point $x = 0$ via Taylor's theorem with Lagranges form of remainder.

Sol. Given $f(x) = e^x \Rightarrow f'(x) = e^x = f''(x) = f'''(x) = \cdots$
By Taylor's theorem, with Lagranges form of remainder,

$$\begin{aligned} e^x = f(x) &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n}{n!}f^n(\theta x), \;\; 0 < \theta < 1 \\ &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}e^{\theta x}, \;\; 0 < \theta < 1 \end{aligned}$$

Expansion of a function as an infinite series. A function $f(x)$ which is defined t $x = a$ and possesses derivative upto nth order at $x = a$, then it can be expressed as an infinite series of the form

$$f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^n(a) + \cdots,$$

if R_n, the remainder of any form, after n terms resulting from Taylor's expansion of $f(x)$ about $x = a$ tends to 0 as $n \to \infty$, i.e, $\lim_{n\to\infty} R_n = 0$.
If$a = 0$, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{(n)!}f^n(0) + \cdots$$

is called the expansion of $f(x)$ is Maclaurins infinite series or expansion of $f(x)$ about $x = 0$.

Example 1.2. Expand e^x as an infinite series.

Sol. Let $f(x) = e^x$, then $f^n(x) = e^x$. From Maclaurin's theorem R_n, the remainder after n terms in Lagrange's form is $\frac{x^n}{n!}e^{\theta x}$, $0 < \theta < 1$.
Note that $0 < \theta < 1 \Rightarrow 0 < \theta x < x \Rightarrow 1 < e^{\theta x} < e^x$. Also for all real x, $e^{\theta x}$ is bounded.
Let $u_n = \frac{x^n}{n!}$, $u_{n+1} = \frac{x^{n+1}}{(n+1)!}$. Then

$$\begin{aligned} \left|\frac{u_{n+1}}{u_n}\right| &\Rightarrow \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = 0 \\ &\Rightarrow \lim_{n\to\infty} u_n = 0 \\ &\Rightarrow \lim_{n\to\infty} R_n = 0 \end{aligned}$$

Therefore,

$$\begin{aligned} f(x) = e^x =& f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots \\ =& 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \end{aligned}$$