# Advanced Engineering Mathematics
# Lecture 48

**Unit tangent vector:** Let $s$ be the arc length and $\hat{t}$ be the unit tangent vector. Then $\frac{d\vec{r}}{ds} = \hat{t}$ and the equation of the tangent line will be

$$(\vec{R} - \vec{r}) \times \hat{t} = \vec{0}.$$

**Unit principle normal vector:** As the length of $\hat{t}$ is constant, its derivative $\frac{d\hat{t}}{ds}$, if not zero, must be perpendicular to $\hat{t}$. Therefore, it will be normal to the curve at the point $P$. A directed line through $P$ in the direction of $\frac{d\hat{t}}{ds}$ is called principle normal vector to the curve $C$ at the point $P$. If $\hat{n}$ is the unit vector in the direction of principle normal vector, then we may write

$$\frac{d\hat{t}}{ds} = \kappa\hat{n},$$

where $\kappa$ is a non-negative scalar and $\hat{n}$ is the unit principle normal. $\kappa$ is the called curvature of the curve at the point $P$.

**Unit binormal vector:** We introduce an another unit normal vector $\hat{b}$ defined by $\hat{b} = \hat{t} \times \hat{n}$, where $\hat{b}$, $\hat{t}$, and $\hat{n}$ forms a right-handed system of orthogonal vectors. A directed line through $P$ in the direction of $\hat{b}$ is called binormal to the curve at the point $P$. We may call $\hat{b}$ as unit binormal at $P$.

**Torsion:** We have defined $\hat{b} = \hat{t} \times \hat{n}$ and since $\hat{b}$ is a unit vector, $\frac{d\hat{b}}{ds}$, if not zero, must be perpendicular to $\hat{b}$, then

$$\begin{aligned}
\hat{b} &= \hat{t} \times \hat{n} \\
\frac{d\hat{b}}{ds} &= \frac{d\hat{t}}{ds} \times \hat{n} + \hat{t} \times \frac{d\hat{n}}{ds} \\
&= \kappa\hat{n} \times \hat{n} + \hat{t} \times \frac{d\hat{n}}{ds} \\
\frac{d\hat{b}}{ds} &= \hat{t} \times \frac{d\hat{n}}{ds}.
\end{aligned}$$

Then $\frac{d\hat{b}}{ds}$ is perpendicular to $\hat{t}$. But it is also perpendicular to $\hat{b}$. Therefore, it must be parallel to $\hat{n}$. Then

$$\frac{d\hat{b}}{ds} = -\tau\hat{n},$$

where $\tau$ is a scalar, is called torsion of the curve at the point $P$. The minus sign is taken because when $\tau$ is positive, $\frac{d\hat{b}}{ds}$ has the direction of $-\hat{n}$ and $P$ moves along with positive direction.

Again,

$$\begin{aligned}
\hat{n} &= \hat{b} \times \hat{t} \\
\frac{d\hat{n}}{ds} &= \frac{d\hat{b}}{ds} \times \hat{t} + \hat{b} \times \frac{d\hat{t}}{ds} \\
&= -\tau\hat{n} \times \hat{t} + \hat{b} \times \kappa\hat{n} \\
&= \tau\hat{b} - \kappa\hat{t}.
\end{aligned}$$

Summarizing the relations we have

(i) $\hat{b} = \hat{t} \times \hat{n}, \quad \hat{n} = \hat{b} \times \hat{t}, \quad \hat{t} = \hat{n} \times \hat{b}.$

(ii) **Serret-Frenet Formula:** $\frac{d\hat{t}}{ds} = \kappa\hat{n}, \quad \frac{d\hat{b}}{ds} = -\tau\hat{n}, \quad \frac{d\hat{n}}{ds} = \tau\hat{b} - \kappa\hat{t}.$

Let $\vec{r} = \vec{f}(t)$ be a vector function.

$$\begin{aligned}
\frac{d\vec{r}}{dt} = \dot{\vec{r}} &= \frac{d\vec{r}}{ds} \cdot \frac{ds}{dt} = \hat{t}\dot{s} \\
|\dot{\vec{r}}| &= |\hat{t}||\dot{s}| = |\dot{s}| \\
\ddot{\vec{r}} &= \frac{d\hat{t}}{ds}\dot{s}^2 + \hat{t}\ddot{s} = \kappa\hat{n}\dot{s}^2 + \hat{t}\ddot{s} \\
\dddot{\vec{r}} &= (\dddot{s} - \kappa^2\dot{s})\hat{t} + (3\dot{s}\ddot{s}\kappa + \dot{s}^2\kappa)\hat{n} + \dot{s}^3\kappa\tau\hat{b}
\end{aligned}$$