Advanced Engineering Mathematics
Lecture 47

Application to Mechanics

Curve in space: A curve is an aggregate of points where coordinates are function of a single variable. Thus the equations

$$x = x(t), \qquad y = y(t), \qquad z = z(t)$$

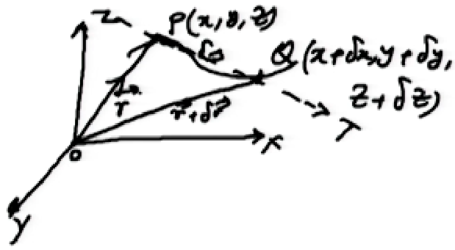
represents a curve in space and the variable "t" is called the parameter. For each value of $t \in \mathbb{R}$ (i.e., $a \le t \le b \in \mathbb{R}$), there corresponds a definite point $P(x, y, z)$ of the curve.

Example 1.

(i) In xy-plane: $x^2 + y^2 = 25$ is a circle. $x = 5\cos t$, and $y = 5\sin t$ where $0 \le t \le 2\pi$.

(ii) In xyz-plane: $x^2+y^2 = 25$ is a sphere. $x = 5\cos\theta\cos\phi$, $y = 5\cos\theta\sin\phi$, and $z = 5\sin\theta$ where $0 \le \phi \le 2\pi$ and $0 \le \theta \le \pi$.

Let $\vec{r} = \vec{f}(t)$ represent the equation of a curve. Now,



$$\begin{aligned}\vec{r} =& x\,\hat{i} + y\,\hat{j} + z\,\hat{k}\\ =& x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}, \qquad t \in \mathbb{R}.\end{aligned}$$

From here, we take $x = x(t)$, $y = y(t)$, and $z = z(t)$. The tangent line PT at a point P of a curve is the limiting position of the secant PQ joining P to a neighboring point Q, i.e., when Q approaches P along the curve.

$$\begin{aligned}\vec{PQ} =& \vec{OQ} - \vec{OP} = \vec{r} + \delta\vec{r} - \vec{r} = \vec{f}(t+\delta t) - \vec{f}(t)\\ \lim_{\delta t\to 0}\frac{\vec{PQ}}{\delta t} =& \lim_{\delta t\to 0}\frac{\delta\vec{r}}{\delta t} = \lim_{\delta t\to 0}\frac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t}\\ \frac{d\vec{r}}{dt} =& \vec{f}\,'(t)\end{aligned}$$

That is, $\frac{d\vec{r}}{dt} = \vec{f}\,'(t)$ is parallel to the tangent PT of the curve $\vec{r} = \vec{f}(t)$ at P. Therefore,

$$\begin{aligned}\vec{R} =& \vec{r} + \lambda\frac{d\vec{r}}{dt}\\ \vec{R} - \vec{r} =& \lambda\frac{d\vec{r}}{dt}, \qquad \text{(Equation of the tangent line)}\\ \text{or, } (\vec{R} - \vec{r}) \times \frac{d\vec{r}}{dt} =& \vec{0}.\end{aligned}$$

where λ is any arbitrary constant, $\vec{r}$ is the position vector of the P and $\vec{R}$ is the position of any point on the tangent line.

Example 2. Find the equation of a tangent line to the space curve $x = t$, $y = t^2$, and $z = \frac{2}{3}t^3$ at $t = 1$.

Solution: We can write $\vec{r}(t) = t\,\hat{i} + t^2\,\hat{j} + \frac{2}{3}t^3\,\hat{k}$. Therefore, the position of the point at $t = 1$ is $\vec{P} = \hat{i} + \hat{j} + \frac{2}{3}\hat{k}$.

$$\begin{aligned} \frac{d\vec{r}}{dt} &= \hat{i} + 2t\,\hat{j} + 2t^2\,\hat{k} \\ \left.\frac{d\vec{r}}{dt}\right|_{t=1} &= \hat{i} + 2\hat{j} + 2\hat{k}. \end{aligned}$$

Hence, the equation of the tangent line

$$\begin{aligned} \big((X\,\hat{i} + Y\,\hat{j} + Z\,\hat{k}) - (\hat{i} + \hat{j} + \frac{2}{3}\hat{k})\big) \times (\hat{i} + 2\hat{j} + 2\hat{k}) &= \vec{0} \\ \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ X-1 & Y-1 & Z-\frac{2}{3} \\ 1 & 2 & 2 \end{vmatrix} &= \vec{0} \\ \big(2Y - 2Z - \frac{2}{3}\big)\hat{i} + \big(Z - 2X + \frac{4}{3}\big)\hat{j} + \big(2X - Y - 1\big)\hat{k} = \vec{0} \end{aligned}$$