Advanced Engineering Mathematics
Lecture 42

1 Differentiation & Integration

Consider a vector-valued function $\vec{r}$ defined on a open interval I containing t_0 and t_1, one can compute the displacement of $\vec{r}$ on $[t_0, t_1]$, see Figure 2(a). Dividing the displacement vector by $t_1 - t_0$ gives the average rate of change of $\vec{r}$ on $[t_0, t_1]$, see Figure 2(b).

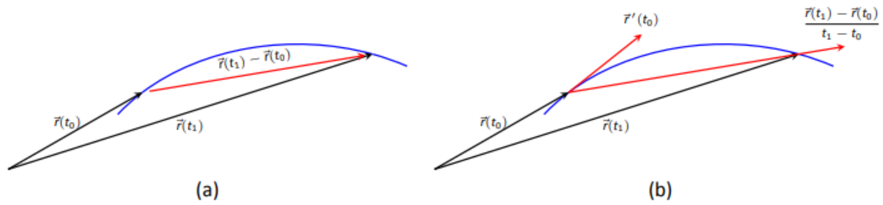


Figure 1: Illustrating displacement, leading to an understanding of the derivative of vector-valued functions.

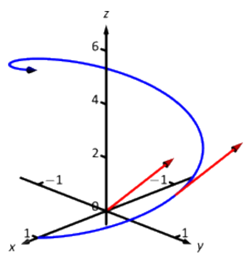
Definition 1. Let $\vec{r}$ be a continuous function on an open interval I containing a. Then the derivative of $\vec{r}$ at $t = a$ is given by

$$\vec{r}\,'(a) = \left.\frac{d\vec{r}}{dt}\right|_{t=a} = \lim_{h \to 0} \frac{\vec{r}(a+h) - \vec{r}(a)}{h}.$$

Let f_1, f_2, f_3 be a differentiable scalar function. If the vector function defined by $\vec{r}(t) = f_1(t)\,\hat{i} + f_2(t)\,\hat{j} + f_3(t)\,\hat{k}$. Then

$$\vec{r}\,'(t) = f_1'(t)\,\hat{i} + f_2'(t)\,\hat{j} + f_3'(t)\,\hat{k}.$$

Example 1. Let $\vec{f}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$. Then $\vec{f}\,'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}$. At $t = \frac{\pi}{2}$, we have $\vec{f}\,'(\frac{\pi}{2}) = -\hat{i} + \hat{k}$.



Properties: Let $\vec{f}$ and $\vec{g}$ be differentiable vector functions, let h be a differentiable scalar function, and let c be a real number.

1. $\frac{d}{dt}\big(\vec{f}(t) \pm \vec{g}(t)\big) = \vec{f}\,'(t) \pm \vec{g}\,'(t)$.

2. $\frac{d}{dt}\big(c\vec{f}(t)\big) = c\vec{f}\,'(t)$.

3. $\frac{d}{dt}\big(h(t)\vec{f}(t)\big) = h'(t)\vec{f}(t) + h(t)\vec{f}\,'(t)$. **Product rule**

4. $\frac{d}{dt}\big(\vec{f}(t) \cdot \vec{g}(t)\big) = \vec{f}\,'(t) \cdot \vec{g}(t) + \vec{f}(t) \cdot \vec{g}\,'(t)$. **Product rule**

5. $\frac{d}{dt}\left(\vec{f}(t)\times\vec{g}(t)\right)=\vec{f}'(t)\times\vec{g}(t)+\vec{f}(t)\times\vec{g}'(t)$. **Product rule**

6. $\frac{d}{dt}\Big(\vec{f}\big(h(t)\big)\Big)=\vec{f}'\big(h(t)\big)h'(t)$. **Chain Rule**

Successive derivative: Let the vector function $\vec{f}$ be n-times differentiable. Then

$$\begin{aligned}
\text{1st Order:}\quad & \tfrac{d\vec{f}}{dt}\\
\text{2nd Order:}\quad & \tfrac{d^2\vec{f}}{dt^2}=\tfrac{d}{dt}\left(\tfrac{d\vec{f}}{dt}\right)\\
&\vdots\\
n\text{th Order}\quad & \tfrac{d^n\vec{f}}{dt^n}=\tfrac{d}{dt}\left(\tfrac{d^{n-1}\vec{f}}{dt^{n-1}}\right)
\end{aligned}$$

Example 2.

1. If a curve is given by $\vec{r}(t)=\vec{a}t+\vec{b}$, then $\frac{d^2\vec{r}}{dt^2}=\vec{0}$.

2. If $\vec{r}(t)=(t+1)\,\hat{i}+(t^2+t+1)\,\hat{j}+(t^3+t^2+t+1)\,\hat{k}$. Then

$$\begin{aligned}
\frac{d\vec{r}}{dt}&=\hat{i}+(2t+1)\,\hat{j}+(3t^2+2t+1)\,\hat{k}\\
\frac{d^2\vec{r}}{dt^2}&=2\,\hat{j}+(6t+2)\,\hat{k}\\
\frac{d^3\vec{r}}{dt^3}&=6\,\hat{k}
\end{aligned}$$

3. If $\vec{r}(t)=\sin t\,\hat{i}+\cos t\,\hat{j}+t\,\hat{k}$. Then

$$\begin{aligned}
\frac{d\vec{r}}{dt}&=\cos t\,\hat{i}-\sin t\,\hat{j}+\,\hat{k}\\
\frac{d^2\vec{r}}{dt^2}&=-\sin t\,\hat{i}-\cos t\,\hat{j}\\
\left|\frac{d\vec{r}}{dt}\right|&=\sqrt{\cos^2 t+\sin^2 t+1}=\sqrt{2}\\
\left|\frac{d^2\vec{r}}{dt^2}\right|&=\sqrt{\sin^2 t+\cos^2 t}=1
\end{aligned}$$

4. Let $\vec{r}(t)=\cos nt\,\hat{i}+\sin nt\,\hat{j}$. Then $\frac{d\vec{r}}{dt}=-n\sin nt\,\hat{i}+n\cos nt\,\hat{j}$.

$$\begin{aligned}
\vec{r}\times\frac{d\vec{r}}{dt}&=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \cos nt & \sin nt & 0\\ -n\sin nt & n\cos nt & 0\end{vmatrix}\\
&=(n\cos^2 nt+n\sin^2 nt)\hat{k}\\
&=n\hat{k}.
\end{aligned}$$

Integration: Let $\vec{f}$ and $\vec{F}$ be two vector functions such that $\frac{d\vec{F}}{dt}=\vec{f}$. Then $\vec{F}$ is called the indefinite integration of $\vec{f}$ with respect to t and we write

$$\int \vec{f}(t)\,dt=\vec{F}(t).$$

Definite integral: $\displaystyle\int_a^b \vec{f}(t)\,dt=\vec{F}(t)\big|_{t=a}^{t=b}=\vec{F}(b)-\vec{F}(a).$

Example 3.

1. Let $\vec{f}(t) = e^{2t}\,\hat{\imath} + \sin t\,\hat{\jmath}$. Evaluate $\displaystyle\int_0^1 \vec{f}(t)\,dt$.

   Solution:

$$\begin{aligned}\int_0^1 \vec{f}(t)\,dt =& \int_0^1 e^{2t}\,dt\,\hat{\imath} + \int_0^1 \sin t\,dt\,\hat{\jmath}\\ =& \frac{1}{2}e^{2t}\Big|_0^1\,\hat{\imath} - \cos t\Big|_0^1\,\hat{\imath}\\ =& \frac{1}{2}(e^2-1)\,\hat{\imath} + (1-\cos 1)\,\hat{\jmath}\end{aligned}$$

2. Let $\vec{f}\,''(t) = 2\,\hat{\imath} + \cos t\,\hat{\jmath} + 12t\,\hat{k}$. Find $\vec{f}$ where $\vec{f}(0) = -7\hat{\imath} - \hat{\jmath} + 2\hat{k}$, and $\vec{f}\,'(0) = 5\hat{\imath} + 3\hat{\jmath}$.

   Solution:

$$\begin{aligned}\vec{f}\,'(t) =& \int \vec{f}\,''(t)\,dt\\ =& 2t\,\hat{\imath} + \sin t\,\hat{\jmath} + 6t^2\,\hat{k} + \vec{a}.\\ \vec{f}\,'(0) =& \vec{a}.\end{aligned}$$

   Therefore, $\vec{f}\,'(t) = (2t+5)\,\hat{\imath} + (\sin t + 3)\,\hat{\jmath} + 6t^2\,\hat{k}$.

$$\begin{aligned}\vec{f}(t) =& \int \vec{f}\,'(t)\,dt\\ =& (t^2+5t)\,\hat{\imath} + (-\cos t + 3t)\,\hat{\jmath} + 2t^3\,\hat{k} + \vec{b}.\\ \vec{f}(0) =& -\hat{\jmath} + \vec{b}\\ \vec{b} =& -7\hat{\imath} + 2\hat{k}.\end{aligned}$$

   Therefore, $\vec{f}(t) = (t^2 + 5t - 7)\hat{\imath} + (-\cos t + 3t)\,\hat{\jmath} + (2t^3 + 2)\,\hat{k}$.