Advanced Engineering Mathematics
Lecture 41

1 Scalar and Vector functions

A function f is a *scalar function* if its maps from $\mathbb{R}^n$ to $\mathbb{R}$, i.e., the range of the function is some subset of real number.

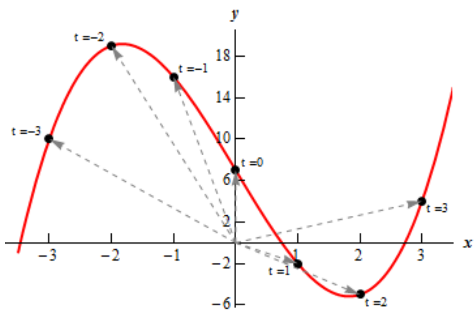
Example 1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by (i) $f(x) = x^2$, (ii) $f(x) = e^x$, (iii) $f(x) = \sin x$.

Definition 1. Let $D \subset \mathbb{R}^n$ be a non-empty set. A function $\vec{f} : D \subset \mathbb{R}^n \to E \subset \mathbb{R}^m$ is called *vector function* if there exist scalar functions $f_1, f_2, \ldots, f_m$ defined on D such that

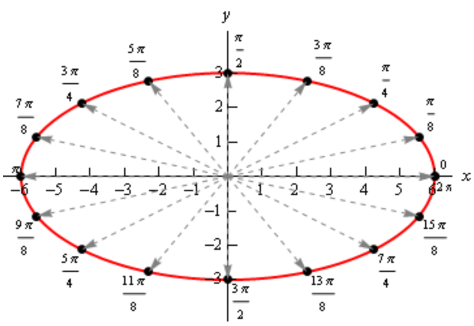
$$\vec{f}(x) = \big(f_1(x), f_2(x), \ldots, f_m(x)\big).$$

Through out the lecture we consider vector functions which are defined on real line, i.e., $n = 1$.

Example 2. Let $\vec{f} : \mathbb{R} \to \mathbb{R}^2$ defined by $\vec{f}(t) = \big(t, t^3 - 10t + 7\big)$. This implies at time t, position of the particle with respect to first coordinate is t and second coordinate is $t^3 - 10 + 7$. The sketch of the vector function is the following.



Example 3. Let $\vec{f} : \mathbb{R} \to \mathbb{R}^2$ defined by $\vec{f}(t) = \big(6\cos t, 3\sin t\big)$. Value of the $\vec{f}$ at time $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ are $(6, 0), (0, 3), (-6, 0), (0, -3)$. The vector function is periodic with the period of 2π. So, in this case the vector function is an ellipse.



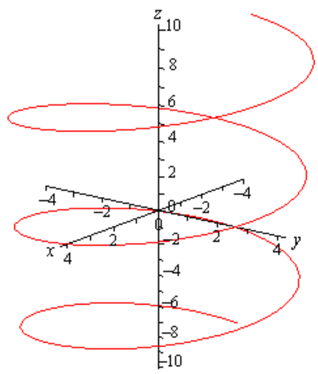
Example 4. The position of a particle on the unit circle at time t can be defined by a vector function $\vec{f}(t) = \big(\cos t, \sin t\big)$. This can also be written as

$$\vec{f}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}.$$

Similarly, for $m = 3$, the vector function $f : \mathbb{R} \to \mathbb{R}^3$ is of the form

$$\begin{aligned}\vec{f}(t) =& \big(4\cos t, 4\sin t, t\big)\\ =& 4\cos t\,\hat{\imath} + 4\sin t\,\hat{\jmath} + t\,\hat{k}.\end{aligned}$$

In this case, along the x and y coordinate function forms a circle. The only change is in the z component and as t increases the z coordinate will increase. So the vector function is forms a spiral.



Scalar field: If to each point $p(x, y, z)$ of a region $\mathcal{R}$ in space there corresponds a unique scalar $f(p)$, then f is called a scalar point function and we say that a scalar field f is defined on $\mathcal{R}$.

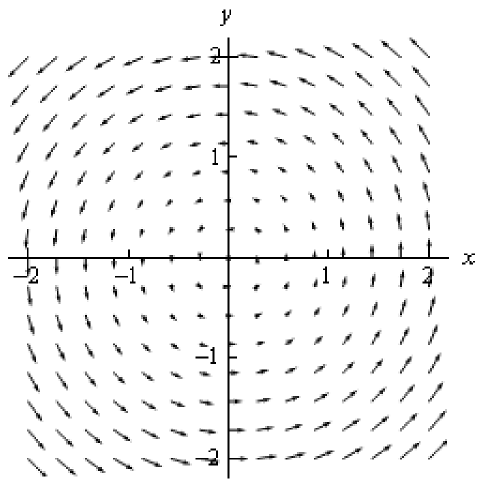
Example: $f(x, y, z) = x^2 + y^2 - 4z^2$ is a scalar field.

Vector field: If to each point $p(x, y, z)$ of a region $\mathcal{S}$ in space there corresponds a unique vector $\vec{f}(p)$, then $\vec{f}$ is called a vector point function and we say that a vector field $\vec{f}$ is defined on $\mathcal{S}$.

Example 5. Let the vector point function $\vec{f}$ on $\mathbb{R}^2$ be defined by

$$\vec{f}(x, y) = -y\,\hat{\imath} + x\,\hat{\jmath}.$$

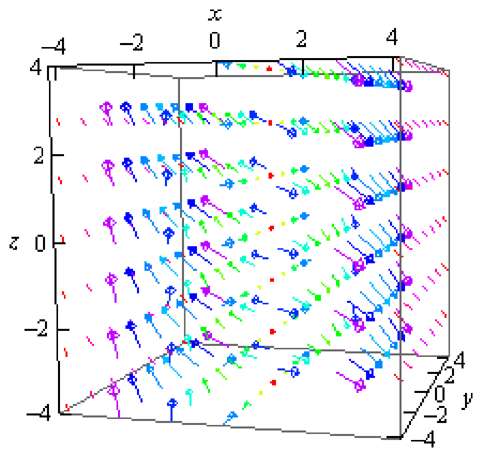
The graph of the vector field is given by the following sketch.

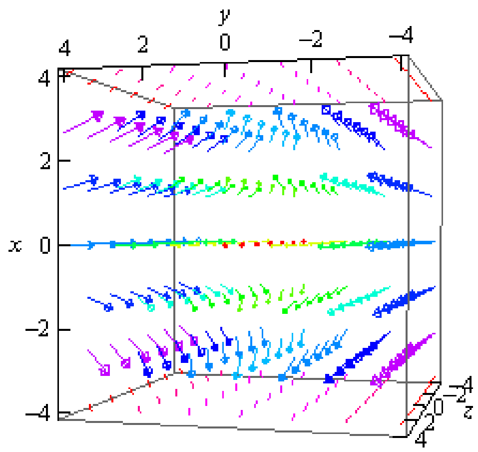


This gives the direction of particle movement at each points in $\mathbb{R}^2$.
Similarly, the vector field of the vector function

$$\vec{f}(x, y, z) = 2x\,\hat{\imath} - 2y\,\hat{\jmath} - 2z\,\hat{k}$$

is given by the following graph.





2 Limit and Continuity

The definitions of limits and continuity for vector functions are simple modifications of the analogous concepts for scalar functions. Using the component functions, we deal with these computations and the idea of continuity in practice.

Definition 2. Let I be an open interval, and let $\vec{f} : I \to \mathbb{R}^m$ be a vector function defined by $\vec{f}(t) = \big(f_1(t), f_2(t), \ldots, f_m(t)\big)$. The limit of $\vec{f}(t)$ is $\vec{\ell} = (\ell_1, \ell_2, \ldots, \ell_m)$ as t approaches to a if for each scalar function f_i, $i = 1, 2, \ldots, m$, $\lim\limits_{t\to a} f_i(t) = \ell_i$. Then we expressed as

$$\lim_{ta} \vec{f}(t) = \Big(\lim_{t\to a} f_1(t), \lim_{t\to a} f_2(t), \ldots, \lim_{t\to a} f_m(t)\Big) = \vec{\ell}.$$

Alternatively, we say $\vec{f}(t)$ approaches to $\vec{\ell}$ as t approaches to a, if for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\vec{f}(t) - \vec{\ell}| < \varepsilon \qquad \text{whenever } 0 < |t - a| < \delta.$$

Properties:

1. $\lim\limits_{t\to a} \big(\vec{f}(t) \pm \vec{g}(t)\big) = \lim\limits_{t\to a} \vec{f}(t) \pm \lim\limits_{t\to a} \vec{g}(t)$.
2. for any $c \in \mathbb{R}$, $\lim\limits_{t\to a} c \cdot \vec{f}(t) = c \cdot \lim\limits_{t\to a} \vec{f}(t)$.
3. $\lim\limits_{t\to a} \big(\vec{f}(t) \cdot \vec{g}(t)\big) = \lim\limits_{t\to a} \vec{f}(t) \cdot \lim\limits_{t\to a} \vec{g}(t)$.
4. $\lim\limits_{t\to a} \big(\vec{f}(t) \times \vec{g}(t)\big) = \lim\limits_{t\to a} \vec{f}(t) \times \lim\limits_{t\to a} \vec{g}(t)$.

Definition 3. Let I be an open interval. A vector function $\vec{f} : I \to \mathbb{R}^m$ is continuous at a if

(i) $\vec{f}(a)$ is defined.

(ii) for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\vec{f}(t) - \vec{f}(a)| < \varepsilon \qquad \text{whenever } |t - a| < \delta.$$

Alternatively, $\vec{f}$ is continuous at a, if

$$\lim_{t\to a} \vec{f}(t) = \vec{f}(a).$$

Example 6. Let $\vec{f}(t) = \left(\frac{\sin t}{t}\,, t^2 - 3t + 3\,, \cos t\right)$.

$$\begin{aligned}\lim_{t\to 0} \vec{f}(t) =& \left(\lim_{t\to 0} \frac{\sin t}{t}\,, \lim_{t\to 0} t^2 - 3t + 3\,, \lim_{t\to 0} \cos t\right) \\ =& (1, 3, 1).\end{aligned}$$

While the second and third components of $\vec{f}$ are defined at $t = 0$, the first component $\frac{\sin t}{t}$ is not. Hence, $\vec{f}$ is not defined at $t = 0$. Therefore, $\vec{f}$ is not continuous at $t = 0$. However, limit of $\vec{f}(t)$ exist as t approaches to 0.

3 Partial Derivative

Suppose $\vec{r}$ is a vector function depending on more than one scalar variable. Let $\vec{r} = \vec{f}(x, y, z)$, i.e., $\vec{r}$ is a function of variable x, y and z. Thus partial derivative of $\vec{r}$ with respect to x is defined by

$$\frac{\partial \vec{r}}{\partial x} = \lim_{h\to 0} \frac{\vec{r}(x + h, y, z) - \vec{r}(x, y, z)}{h}.$$

Similarly, we can defined $\frac{\partial \vec{r}}{\partial y}, \frac{\partial \vec{r}}{\partial z}, \frac{\partial^2 \vec{r}}{\partial x \partial y}, \ldots$.

Vector Differential Operator: The vector differential operator $\vec{\nabla}$ (nabla) and it is defined as

$$\vec{\nabla} = \frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\,\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}.$$

Gradient of a scalar function: Let f be defined and differentiable at each point (x, y, z) in a certain region of space. Then the gradient of f is denoted by

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\,\hat{\imath} + \frac{\partial f}{\partial y}\,\hat{\jmath} + \frac{\partial f}{\partial z}\,\hat{k}.$$

Example 7. Let the scalar function $f(x, y, x) = x^3yz^2$. Then the gradient of f is

$$\begin{aligned}\vec{\nabla} f(x, y.z) =& 3x^2yz^2\,\hat{\imath} + x^3z^2\,\hat{\jmath} + 2x^3yz\,\hat{k} \\ \vec{\nabla} f(1, 1, 2) =& 12\hat{\imath} + 4\hat{\jmath} + 4\hat{k}.\end{aligned}$$

Properties: Let f and g be a multi-variable scalar function.

1. $\vec{\nabla}(f \pm g) = \vec{\nabla} f \pm \vec{\nabla} g$
2. $\vec{\nabla}(fg) = g\vec{\nabla} f + f\vec{\nabla} g$

Definition 4. Let $f(x, y, z)$ be a scalar field over a region $\mathcal{R}$. Then the points satisfying an equation of the type

$$f(x, y, z) = c$$

constitute a family of surface in 3-dimensional space is called *level surface.*

Lemma 3.1. *Let f be a scalar function. Then $\vec{\nabla} f$ is a vector normal to the surface $f(x, y, z) = c$, where c is a constant.*