Advanced Engineering Mathematics
Lecture 4

1 Higher Order Derivatives

$$
\begin{aligned}
y = f(x) \Rightarrow \frac{dy}{dx} &= \frac{d}{dx}(f(x)) = y_1 = y' \\
\frac{d^2y}{dx^2} &= \frac{d}{dy}\Big(\frac{dy}{dx}\Big) = \frac{d}{dx}\Big(\frac{df}{dx}\Big) = y_2 = y'' \\
\frac{d^3y}{dx^3} &= \frac{d^2}{dx^2}\Big(\frac{df}{dx}\Big) = y_3 = y''' \\
&\vdots \\
\frac{d^ny}{dx^n} &= \frac{d^{n-1}}{dx^{n-1}}\Big(\frac{df}{dx}\Big) = y_n = y^n, \quad n \in \mathbb{N}
\end{aligned}
$$

Sometimes, $y_n = y^n$ is denoted by $y^{(n)}$ or D^n where $D = \frac{d}{dx}$.

Example 1.1. Let $y = f(x) = x^n$, $n \in \mathbb{N}$. Then find $D^m y = \frac{d^m y}{dx^m}$.

Sol.

$$
\begin{aligned}
&Dy = nx^{n-1} \\
&D^2y = d(dy) = n(n-1)x^{n-2} \\
&\quad\vdots \\
&D^my = n(n-1)\cdots(n-m+1)x^{n-m},
\end{aligned}
$$

when $m = n$, then $D^my = n(n-1)\cdots 1 \;\; x^0 = n!$

Example 1.2. Let $y = \log_e(ax+b)$, $a, b > 0$

Sol.

$$
\begin{aligned}
y_1 = Dy &= \frac{1}{ax+b}a = \frac{a}{ax+b} \\
y_2 &= -\frac{a^2}{(ax+b)^2} \\
&\vdots \\
y_n = D^ny &= d^{n-1}y_n = ad^{n-1}\Big(\frac{1}{ax+b}\Big) \\
&= \frac{a^n(-1)^{n-1}(n-1)!}{(ax+b)^n} \\
\Rightarrow \;\; y_n &= \frac{a^n(-1)^{n-1}(n-1)!}{(ax+b)^n}
\end{aligned}
$$

Theorem 1.1. *(Leibnitz Rule Differentiation for Product of Functions) If u and v be two functions of x defined on an interval $[a, b]$ and each is differentiable n times w,r,t, "x", the $n-th$ derivative of the product is given by*

$$(uv)_n = D^n(uv) = u_nv + {}^nC_1u_{n-1}v_1 + {}^nC_2u_{n-2}v_2 + \cdots + {}^nC_ru_{n-r}v_r + \cdots + uv_n$$

where the suffixes denote the order of derivative.

Example 1.3. If $y = \cos(m\sin^{-1}x)$, then show that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$$

Soln.

$$\begin{aligned} & y = \cos(m\sin^{-1}x) \Rightarrow y_1 = Dy = -\sin(m\sin^{-1}x)\frac{m}{\sqrt{1-x^2}} \\ \Rightarrow\ & (1-x^2)y_1^2 = m^2\sin^2(m\sin^{-1}x) = m^2[1-\cos^2(m\sin^{-1}x)] = m^2(1-y^2) \\ & \text{(differentiating w.r.t. } x) \\ \Rightarrow\ & (1-x^2)^2 y_1y_2 - 2xy_1^2 = -2m^2yy_1 \\ \Rightarrow\ & (1-x^2)y_2 - xy_1 + m^2y = 0 \end{aligned}$$

Differentiating w.r.t. x on both sides n times,

$$(1-x^2)y_{n+2} + {}^nC_1y_{n+1}(-2x) + {}^nC_2y_n(-2) + {}^nC_3y_{n-1}(0) + \cdots - xy_{n+1} - {}^nC_1y_n - \cdots + m^2y_n = 0$$
$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$$

Example 1.4. If $y = \frac{\log x}{x}$, then prove that $y_n = \frac{(-1)^n n!}{x^{n+1}}\Big[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n}\Big]$.

Sol. Let us take $u = \frac{1}{x}$ and $v = \log x$. Then

$$y_n = (uv)_n = u_nv + \sum_{r=1}^{n} {}^nC_r u_{n-r}v_r$$

Here, $u_n = \frac{(-1)^n n!}{x^n}$, $u_{n-r} = \frac{(-1)^{n-r}(n-r)!}{x^{n-r+1}}$ and $v_r = \frac{(-1)^{r-1}(r-1)!}{x^r}$ Now

$$\begin{aligned} y_n &= (uv)_n = u_nv + \sum_{r=1}^{n} {}^nC_r u_{n-r}v_r \\ &= u_n = \frac{(-1)^n n!}{x^{n+1}}\log x + \sum_{r=1}^{n} {}^nC_r \frac{(-1)^{n-r}(n-r)!}{x^{n-r+1}} \cdot \frac{(-1)^{r-1}(r-1)!}{x^r} \\ &= \frac{(-1)^n n!}{x^{n+1}}\Big[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n}\Big] \end{aligned}$$