Advanced Engineering Mathematics
Lecture 39

Orthogonal Matrix

Let $A = (a_{ij})_{n\times n}$ be a square matrix. Then, A is said to be an Orthogonal Matrix, if $AA^T = I_n$.

Proposition 1. *If A is orthogonal, then A is non-singular.*
We can see $AA^T = I_n \implies |A||A^T| = 1 \implies |A|^2 = 1$. Hence, $\det(A) \neq 0$.

Proposition 2. *If A is orthogonal, then $A^{-1} = A^T$.*

Proposition 3. *If A is orthogonal, then A^{-1} is also orthogonal.*

Complex Matrix

A matrix A, whose elements are taken from $x \in \mathbb{C}$, is called a complex matrix. This means that A can be expressed as $P + \iota Q$, where P and Q are real matrices. Then, $A = P + \iota Q$, $\bar{A} = P - \iota Q$. Some properties of the complex matrices are given below.

(i) $A = \bar{\bar{A}}$.

(ii) A and B are two complex matrices, then $\overline{AB} = \bar{A}\bar{B}$.

(iii) For a complex matrix, $(\bar{A})^T = (\overline{A^T})$. The transpose of $\bar{A}$ is denoted by A°, $(\bar{A})^T = A^\circ$.

Hermitian and Skew-Hermitian Matrices

A complex matrix is said to be Hermitian Matrix if $A = A^\circ$, $A^\circ = (\bar{A})^T$. A complex matrix is said to be a Skew-Hermitian Matrix if $A = -A^\circ$, $A^\circ = -(\bar{A})^T$.

Example 1. Check the matrix $A = \begin{bmatrix} 1 & 2 & 3\iota \\ 2 & 0 & 4 \\ -3\iota & 4 & 2 \end{bmatrix}$ is Hermitian or Skew-Hermitian.

Solution. We get $\bar{A} = \begin{bmatrix} 1 & 2 & -3\iota \\ 2 & 0 & 4 \\ 3\iota & 4 & 2 \end{bmatrix}$ and then $(\bar{A})^T = \begin{bmatrix} 1 & 2 & 3\iota \\ 2 & 0 & 4 \\ -3\iota & 4 & 2 \end{bmatrix} = A$. Thus, A is a Hermitian matrix.

Example 2. Check the matrix $A = \begin{bmatrix} 0 & 2 & \iota \\ -2 & 0 & 1+\iota \\ \iota & -1+\iota & 0 \end{bmatrix}$ is Hermitian or Skew-Hermitian.

Solution. We get $\bar{A} = \begin{bmatrix} 0 & 2 & -\iota \\ -2 & 0 & 1-\iota \\ -\iota & -1-\iota & 0 \end{bmatrix}$ and then $(\bar{A})^T = \begin{bmatrix} 0 & -2 & -\iota \\ 2 & 0 & -1-\iota \\ -\iota & 1-\iota & 0 \end{bmatrix} = -A$. Thus, A is a Skew-Hermitian matrix.

Unitary Matrix

A matrix A is said to be a Unitary Matrix if $A\bar{A}^T = I$.

Example 1. Let $A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$, where $\omega = \frac{-1 \pm \sqrt{3}\iota}{2}$ and the relations $1 + \omega + \omega^2 = 0$, $\omega^3 = 1$ hold. Check whether A is Unitary matrix or not.

Theorem 1. *If $H = P + \iota Q$ be a Hermitian matrix, then*

(i) the diagonal elements of H are all real numbers.

(ii) P is a real Symmetric matrix and Q is a real Skew-Symmetric matrix.

Theorem 2. *If $S = M + \iota N$ be a Skew-Hermitian matrix, then*

(i) the diagonal elements of S are either 0 or imaginary numbers.

(ii) M is a real Skew-Symmetric matrix and N is a real Symmetric matrix.