Advanced Engineering Mathematics
Lecture 37

Example 8. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A.

Solution. Let $\lambda \in \mathbb{F}$. Then, the characteristic equation is: $\det(A - \lambda I) = 0 \implies \begin{vmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix} \implies \lambda^2 + 1 = 0 \implies \lambda = \pm\iota$.

Case 1. Let $\lambda = \iota$ and $X = (x_1, x_2) \in \mathbb{R}^2$. Then

$$\begin{aligned} & AX = \lambda X \\ \implies & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \iota \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \implies & \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} \iota x_1 \\ \iota x_2 \end{bmatrix} \\ \implies & \begin{bmatrix} -\iota x_1 - x_2 \\ x_1 - \iota x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

For eigenvalue $\lambda = \iota$, we get $x_1 = \iota c$, where $x_2 = c \in \mathbb{R}$. Then, the required eigenvector is $X = \begin{bmatrix} \iota \\ 1 \end{bmatrix} c\ ;\ c \in \mathbb{F}$.

Case 2. Let $\lambda = -\iota$ and $X = (x_1, x_2) \in \mathbb{R}^2$. Then

$$\begin{aligned} & AX = \lambda X \\ \implies & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\iota \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \implies & \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} = -\begin{bmatrix} \iota x_1 \\ \iota x_2 \end{bmatrix} \\ \implies & \begin{bmatrix} \iota x_1 - x_2 \\ x_1 + \iota x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

For eigenvalue $\lambda = -\iota$, we get $x_1 = -\iota c$, where $x_2 = c \in \mathbb{R}$. Then, the required eigenvector is $X = \begin{bmatrix} -\iota \\ 1 \end{bmatrix} c\ ;\ c \in \mathbb{F}$.

Diagonalisation of a Matrix

Similar Matrices. Let us consider all $n \times n$ matrices over the field $\mathbb{F}$. An $n \times n$ matrix A is said to be similar to an $n \times n$ matrix B, if there exists a non-singular matrix P of order $n \times n$ such that $B = P^{-1}AP$. We say that A and B are similar matrices.

$$\begin{aligned} & B = P^{-1}AP \\ \implies & PB = PP^{-1}AP = (PP^{-1})AP = IAP = AP \\ \implies & PBP^{-1} = APP^{-1} = A \\ \implies & A = Q^{-1}BQ, \quad \text{where } Q = P^{-1}. \end{aligned}$$

Let $A = (a_{ij})_{n\times n}$ be a square matrix. Then, A is said to be Diagonalisable, if A is similar to an $n \times n$ diagonal matrix.

Example 1. Diagonalise the matrix $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

Solution. *Step 1.* Let $\lambda \in \mathbb{F}$ be the eigenvalue. Then

$$\begin{aligned} &\det(A - \lambda I) = 0 \\ \implies &\begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = 0 \\ \implies &\lambda = -2, 4, -2. \end{aligned}$$

Step 2. Let $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ be the eigenvector.

$$\begin{aligned} &AX = \lambda X \\ \implies &\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ \implies &\begin{bmatrix} x_1 - 3x_2 + 3x_3 \\ 3x_1 - 5x_2 + 3x_3 \\ 6x_1 - 6x_2 + 4x_3 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ \implies &\begin{bmatrix} 3x_1 - 3x_2 + 3x_3 \\ 3x_1 - 3x_2 + 3x_3 \\ 6x_1 - 6x_2 + 6x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Which gives us $x_1 - x_2 + x_3 = 0$. So, $x_2 = c$, $x_3 = d$, $x_1 = c - d$, where $c, d \in \mathbb{F}$. The required solution: $X = (c - d, c, d)$. The eigenvector corresponding to $\lambda = -2$ is $(1, 1, 0)$ and $(-1, 0, 1)$, i.e., $\left\{ \begin{bmatrix} c-d \\ c \\ d \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}; c, d \in \mathbb{F} \right\}$.

Step 3. For $\lambda = 4$, follow the same process and find the eigenvector. it will be continued in the next lecture.