Advanced Engineering Mathematics
Lecture 35

Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix and $\lambda \in \mathbb{F}$. Then, $\det(A - \lambda I)$ is said to be a Characteristic Polynomial of A where $I_{n\times n}$ is the identity matrix, and is denoted by $\Psi_\lambda(A)$. Furthermore, $\Psi_\lambda(A) = 0$ is called the Characteristic Equation. The roots of $\Psi_\lambda(A) = 0$ are called the Eigenvalues of the matrix A. If a root of $\Psi_\lambda(A) = 0$ has multiplicity r, then it is said to be an r-fold eigenvalue. For an r-fold eigenvalue λ, r is called the Algebraic Multiplicity of λ, and the rank of characteristic subspace corresponding to λ is called the Geometric Multiplicity.

Example 1. Find the eigenvalue of the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution. Let $\lambda \in \mathbb{F}$. Then, the characteristic polynomial is, $\det(A - \lambda I) = \begin{vmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = \lambda^2 - 1$. Therefore, the characteristic equation is given by $\det(A - \lambda I) = 0 \implies \lambda^2 - 1 = 0 \implies \lambda = -1, 1$. The required eigenvalues are -1 and 1.

Example 2. Find the eigenvalues of $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution. Let $\lambda \in \mathbb{F}$. Then, the characteristic polynomial is given by

$$\Psi_\lambda(A) = \det(A - \lambda I) = \det\left\{\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\} = \begin{vmatrix} 1-\lambda & 1 & 1 \\ -1 & -1-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{vmatrix}.$$

The characteristic equation is $\Psi_\lambda(A) = 0 \implies \begin{vmatrix} 1-\lambda & 1 & 1 \\ -1 & -1-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$. After solving the determinant, we get $(1-\lambda)\lambda^2 = 0 \implies \lambda = 0, 0, 1$. Clearly, 0 is an eigenvalue with the algebraic multiplicity 2 and 1 has the algebraic multiplicity 1.

Now let $\lambda = \lambda_1$ be an eigenvalue of $A_{n\times n}$. Then, there exists a non-zero vector $X \in \mathbb{R}^n$ such that $AX = \lambda_1 X$.

Example 3. Find the eigenvectors corresponding to the eigenvalues we found in the previous example.

Solution. We are given $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. We know that $\lambda = 0, 0, 1$. Let $X = (x, y, z) \in \mathbb{R}^3$

be the eigenvector corresponding to $\lambda = 0$. Then,

$$\begin{aligned} AX &= \lambda X \\ \implies \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= 0 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ \implies \begin{bmatrix} x+y+z \\ -x-y-z \\ z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

We get: $z = 0$, $x + y = 0 \implies x = -y = -k$, where $k \in \mathbb{R}$. So, the required eigenvector corresponding to the eigenvalue 0 is: $X = k(-1, 1, 0)$, $k \in \mathbb{R}$.

Now let $\lambda = 1$. Then, the eigenvector can be obtain

$$\begin{aligned} AX &= \lambda X \\ \implies \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= 1 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ \implies \begin{bmatrix} x+y+z \\ -x-y-z \\ z \end{bmatrix} &= \begin{bmatrix} x \\ y \\ z \end{bmatrix} \end{aligned}$$

We get: $y + z = 0$, $x + y = -(y + z) = 0$. So, we get $x = z = k \in \mathbb{R}$. The required eigenvector corresponding to the eigenvalue 1 is: $X = k(1, -1, 1)$, $k \in \mathbb{R}$.