Advanced Engineering Mathematics
Lecture 33

continued from *Lecture 32...*
Let us choose one of basis of $\mathbb{R}$ having the elements: $\alpha_1 = (1,0,0), \alpha_2 = (0,1,0), \alpha_3 = (0,0,1)$. From which one can easily get: $T(1,0,0) = (1,2,1) = T(\alpha_1)$, $T(0,1,0) = (1,1,2) = T(\alpha_2)$, $T(0,0,1) = (1,2,1) = T(\alpha_3)$. Clearly, the set $\{T(\alpha_1), T(\alpha_2), T(\alpha_3)\}$ is linearly dependent. Then, we consider the set $\{T(\alpha_1), T(\alpha_2)\} = \{(1,2,1),(1,1,2)\}$, which is linearly independent. The image set $\mathrm{Im}(T)$ is generated/spanned by $\{T(\alpha_1), T(\alpha_2)\}$, i.e., $\mathrm{Im}(T) = \{T(\alpha_1), T(\alpha_2)\}$. Also, $\dim(\mathrm{Im}(T)) = 2$.
From here, we observe that: $\dim(\ker(T)) + \dim(\mathrm{Im}(T)) = 1 + 2 = 3 = \dim\left(\mathbb{R}^3\right) \implies \mathrm{null}(T) + \mathrm{rank}(T) = \dim(V)$.

Theorem 1. *If $T : V \to W$ is a linear transformation, then null$(T) + \mathrm{rank}(T) = \dim(V)$. This theorem is known as Rank-Nullity Theorem.*

Theorem 2. *Let V and W be two vector spaces over a field $\mathbb{F}$, and $T : V \to W$ be a linear transformation. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of V, then $\{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)\}$ generates $\mathrm{Im}(T)$.*

Example 2. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be such that $T(x_1, x_2, x_3) = (x_2 + x_3, x_3 + x_1, x_1 + x_2, x_1 + x_2 + x_3)$.

(i) Verify T is linear.

(ii) Find $\ker(T)$.

(iii) Verify $\{T(1,0,0), T(0,1,0), T(0,0,1)\}$ is linearly independent set.

Solution. (i) Let us consider $\alpha = (x_1, x_2, x_3)$ and $\beta = (y_1, y_2, y_3)$. So,

$$\begin{aligned} T(\alpha + \beta) &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ &= (x_2 + y_2 + x_3 + y_3, x_3 + y_3 + x_1 + y_1, x_1 + y_1 + x_2 + y_2, x_1 + y_1 + x_2 + y_2 + x_3 + y_3) \\ &= (x_2 + x_3 + y_2 + y_3, x_3 + x_1 + y_3 + y_1, x_1 + x_2 + y_1 + y_2, x_1 + x_2 + x_3 + y_1 + y_2 + y_3) \\ &= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) = T(\alpha) + T(\beta) \end{aligned}$$

Similarly, one can check that $T(c\alpha) = cT(\alpha)$. $\implies T$ is linear.
(ii) By definition, $\ker(T) = \left\{\alpha = (x_1, x_2, x_3) \in \mathbb{R}^3 : T(\alpha) = \vec{0} \in \mathbb{R}^4\right\}$. This leads to a system of equations: $x_2 + x_3 = 0$, $x_3 + x_1 = 0$, $x_1 + x_2 = 0$, $x_1 + x_2 + x_3 = 0$, which has the solution: $x_1 = 0, x_2 = 0, x_3 = 0$. Therefore, $\ker(T) = \{(0,0,0)\}$ $\{\theta\}$.
(iii) Now let $\alpha_1 = (1,0,0), \alpha_2 = (0,1,0), \alpha_3 = (0,0,1)$. Then, $T(1,0,0) = (0,1,1,1), T(0,1,0) = (1,0,1,1), T(0,0,1) = (1,1,0,1)$. Let $c_1, c_2, c_3 \in \mathbb{F}$ be such that

$$\begin{aligned} c_1 T(\alpha_1) + c_2 T(\alpha_2) + c_3 T(\alpha_3) &= \vec{0} \\ \implies c_1(0,1,1,1) + c_2(1,0,1,1) + c_3(1,1,0,1) &= (0,0,0,0) \end{aligned}$$

leading to: $c_2 + c_3 = 0$, $c_1 + c_3 = 0$, $c_1 + c_2 = 0$, $c_1 + c_2 + c_3 = 0$. So, $c_1 = 0, c_2 = 0, c_3 = 0$. Clearly, all c_i's, $i = 1,2,3$ are zero. The set $\{T(\alpha_1), T(\alpha_2), T(\alpha_3)\}$ is linearly independent.

Example 3. Let us consider a map $T : \mathbb{R}^3 \to \mathbb{R}^2$ be such that $T(x_1, x_2, x_3) = (x_1 + x_2, x_3 + x_2)$.

(i) Verify if T is linear.

(ii) If yes, find $\ker(T)$ and $\mathrm{Im}(T)$.

Example 4. Let $T : \mathbb{R}^7 \to \mathbb{R}^8$ be such that $T(x_1, x_2, \dots, x_7) = (x_1+x_2, x_2+x_3, \dots, x_7+x_8, x_8+x_1)$.

(i) Verify if T is linear.

(ii) If yes, find $\ker(T)$ and $\mathrm{Im}(T)$.