Advanced Engineering Mathematics
Lecture 31

Example 2. Find the rank of the matrix given below.

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 8 & 6 \\ 0 & 0 & 5 & 8 \\ 3 & 6 & 6 & 3 \end{bmatrix}_{4\times 4}$$

Solution.

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 8 & 6 \\ 0 & 0 & 5 & 8 \\ 3 & 6 & 6 & 3 \end{bmatrix} \xrightarrow[R_2-2R_1]{R_4-3R_1} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 5 & 8 \\ 0 & 0 & 3 & 3 \end{bmatrix} \xrightarrow[\frac{1}{6}R_2]{\frac{1}{3}R_4} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 5 & 8 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[R_4-R_2]{R_3-5R_2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3}R_3} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2-R_3} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B, \text{ (say)}.$$

B is a row-reduced/row-equivalent matrix to A. B is also a row-reduced Echelon matrix to A. Hence, the rank of A is same as B, i.e., $\text{rank}(A) = 3$.

Theorem 1. *If a matrix A is row-equivalent to a row-reduced Echelon matrix/form B having r non-zero rows, then* $\text{rank}(A) = r$.

Example 1 (Inverse of a matrix). Find the inverse of $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{bmatrix}$.

Solution. We form a 3×6 matrix of type $[A/I_3]$ and perform row operations to reduce A to a row-reduced Echelon matrix.

$$B = [A/I_3] = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 4 & 4 & 0 & 1 & 0 \\ 3 & 3 & 7 & 0 & 0 & 1 \end{bmatrix}_{3\times 6} \xrightarrow[R_2-2R_1]{R_3-3R_1} \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1-R_2} \begin{bmatrix} 1 & 0 & 2 & 2 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{bmatrix} \xrightarrow{R_1-2R_3} \begin{bmatrix} 1 & 0 & 0 & 8 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{bmatrix}$$

Hence, the inverse of the matrix A is $C = A^{-1} = \begin{bmatrix} 8 & -\frac{1}{2} & 0 \\ -1 & \frac{1}{2} & 0 \\ -3 & 0 & 1 \end{bmatrix}$.

Example 2. Solve the following system of linear equations.

$$\begin{aligned} x_1 + x_2 &= 4 \\ x_2 - x_3 &= 1 \\ 2x_1 + x_2 + 4x_3 &= 7 \end{aligned}$$

Solution. We can write the above system in the below compact form:

$$A\vec{x} = \vec{b},$$

where $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 4 \end{bmatrix}$, $\vec{x} = (x_1, x_2, x_3)$ and $\vec{b} = (4, 1, 7)$.
Let us form the Augmented matrix:

$$\overline{A} = [A/b] = \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & -1 & 1 \\ 2 & 1 & 4 & 7 \end{bmatrix}_{3\times 4}$$

The steps $R_1 - R_2, R_3 + R_2, \frac{1}{3}R_3, R_1 - R_3, R_2 + R_3$ once applied on A, leads to

$$\overline{A} \equiv \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = C$$

The modified system of equations will be $x_1 = 3, x_2 = 1, x_3 = 0$. Therefore, $(3, 1, 0)$ is a solution of the given system.

Example 3. Verify if the below system has a solution.

$$\begin{aligned} x + 2y - z &= 10 \\ -x + y + 2z &= 2 \\ 2x + y - 3z &= 2 \end{aligned}$$

Solution. The augment matrix,

$$\overline{A} = [A/b] = \begin{bmatrix} 1 & 2 & -1 & 10 \\ -1 & 1 & 2 & 2 \\ 2 & 1 & -3 & 2 \end{bmatrix}$$

The steps $R_2 + R_1, R_3 - 2R_1, \frac{1}{3}R_2, R_1 - 2R_2, R_3 + 3R_2$ once applied on A, leads to

$$\overline{A} \equiv \begin{bmatrix} 1 & 0 & -\frac{5}{3} & 2 \\ 0 & 1 & \frac{1}{3} & 4 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

The modified system of equations:

$$\begin{aligned} x - \frac{5}{3}z &= 2 \\ y + \frac{z}{3} &= 4 \\ 0 &= -6 \text{ (ABSURD)} \end{aligned}$$

Hence, the given system has NO solution.

Gauss-Elimination Method

Example 1. Solve the following system by Gauss-Elimination method:

$$
\begin{aligned}
2x - 2y &= -6 \\
x - y + z &= 1 \\
3y - 2z &= -5
\end{aligned}
$$

Solution. The coefficient matrix, $A = \begin{bmatrix} 2 & -2 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & -2 \end{bmatrix}$, $\vec{x} = (x, y, z)$ and $\vec{b} = (-6, 1, -5)$. The augmented matrix:

$$\overline{A} = [A/b] = \begin{bmatrix} 2 & -2 & 0 & -6 \\ 1 & -1 & 1 & 1 \\ 0 & 3 & -2 & -5 \end{bmatrix}$$

Which reduces after applying the operations $\frac{1}{2}R_1, R_2 - R_1, R_2 \leftrightarrow R_3$ to $\tilde{A} = \begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & 4 \end{bmatrix}$.
From the third row, $z = 4$. Back substituting this value in the equation corresponding to 2nd row leads to, $3y - 2z = -5 \implies y = 1$. Again. we use these values and substitute in the equation corresponding to 1st row and get, $x - y = -3 \implies x = -2$. Hence, the required solution is $(-2, 1, 4)$.