Advanced Engineering Mathematics
Lecture 30

Rank of a Matrix

Let $A = (a_{ij})$, where $1 \leq i, j \leq n$, be a non-zero matrix. Then, the rank of A is said to be $r \in \mathbb{N}$ such that there exists at least one minor (non-zero) of A of order r.

Example 1. Let us consider the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Find the rank of A.

Solution. We notice that $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$. So, $\text{rank}(A) = 3$.

Example 2. Let us consider the matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 5 \\ 2 & 0 & 6 \end{bmatrix}$. Find the rank of A.

Solution. We find that $\det(A) = \begin{vmatrix} 1 & 0 & 3 \\ 4 & -1 & 5 \\ 2 & 0 & 6 \end{vmatrix} = 0$, which implies that $\text{rank}(A) < 3$. Next, we have the minor of order 2, $M = \begin{vmatrix} 1 & 0 \\ 4 & -1 \end{vmatrix} = -1 \neq 0$. Hence, $\text{rank}(A) = 2$.

Example 3. Let us consider the matrix $A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 3 & 0 & 4 & 2 \\ 6 & 9 & -3 & 3 \end{bmatrix}_{3\times 4}$. Find the rank of A.

Solution. Since the third row is a multiple of first row, so the rank of A is always less than 3. We take a minor of order 2, $M = \begin{vmatrix} 2 & 3 \\ 3 & 0 \end{vmatrix} = -9 \neq 0 \implies \text{rank}(A) = 2$.

Note. *If we have $A = (a_{ij})$, where $1 \leq i \leq m, 1 \leq j \leq n$, then $0 < \text{rank}(A) \leq \min(m, n)$.*

Row Equivalent or Column Equivalent Matrices

A matrix B is said to be row equivalent to a matrix A, if it is obtained from A by some successive row operations (finite in number). Similarly, we can define column equivalent matrix.

Example 1. Find the rank of $A = \begin{bmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \end{bmatrix}_{3\times 5}$.

Solution. We start applying row-operations on the matrix A as follows:

$$A = \begin{bmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \end{bmatrix} = B_1$$

$$B_1 \xrightarrow[R_3-2R_1]{R_2-2R_1} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 0 & 2 & 1 \\ 2 & 6 & 0 & 4 & 2 \end{bmatrix} = B_2 \xrightarrow{R_3-2R_2} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B_3 = B.$$

All the matrices B, B_1, B_2, B_3 are row equivalent to A. Note that $\operatorname{rank}(B) = \operatorname{rank}(A)$ as B has a minor of order 2 such that $\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2 \neq 0$.

Row-Reduced or Echelon Form

An $m \times n$ matrix A is called row-reduced (Echelon) matrix, if

(i) the first non-zero element in a non-zero row is 1 (identity element).

(ii) there is an integer r $(0 \leq r \leq m)$ such that the first r rows of A are non-zero rows and the remaining rows (if there is any) are all zero.

(iii) for each non-zero row, if the leading 1 of the row i occurs in the column k_i, then $k_1 < k_2 < \ldots$, and so on.

Example 1. Apply the elementary row operations to reduce the following matrix to a row reduced Echelon matrix.

$$A = \begin{bmatrix} 2 & 0 & 4 & 2 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{bmatrix}_{4\times 4}$$

Solution. Row operations to be performed to get the Echelon form

$$B = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4}$$

of the matrix A are (respectively): $\frac{1}{2}R_1, R_2-3R_1, R_3-5R_1, \frac{1}{2}R_2, R_3-2R_2, R_4-3R_2, R_{34}$ $(R_3 \leftrightarrow R_4)$, $\frac{1}{2}R_3, R_1 - 2R_3$. Clearly, the rank of B is 3 and so is $\operatorname{rank}(A)$.