Advanced Engineering Mathematics
Lecture 28

Vector Space

Let U and S be two non-empty sets. A mapping $f : U \times S \mapsto S$ is said to be an external composition of U with S. Each ordered pair $(u, s) \in U \times S$ has a definite image $f(a, s) \in S$. For example, let S be the set of all real matrices of 3×3 type, and U be the set of all real numbers $U = \mathbb{R}$. The mapping $* : U \times S \mapsto S$ defined by $C * A = CA \in S, \quad C \in \mathbb{R}, A \in S$.
Let $V \neq \emptyset$ and $\bigoplus$ be a composition on V. Let $(\mathbb{F}, +, \cdot)$ be the field of scalars, and let $\bigodot$ be an external composition of $\mathbb{F}$ with V. Then, V is said to be vector space over the field $\mathbb{F}$, if the following conditions are true:

(i) $\alpha \bigoplus \beta \in V$, $\forall \alpha, \beta \in V$ (closedness property).

(ii) $\alpha \bigoplus \beta = \beta \bigoplus \alpha$ for $\alpha, \beta \in V$.

(iii) $(\alpha \bigoplus \beta) \bigoplus \gamma = \alpha \bigoplus (\beta \bigoplus \gamma)$, $\forall \alpha, \beta, \gamma \in V$.

(iv) $\exists$ an element $\theta \in V$ such that $\alpha \bigoplus \theta = \alpha$, $\forall \alpha \in V$.

(v) for each $\alpha \in V$, $\exists (-\alpha) \in V$ such that $\alpha + (-\alpha) = \theta$.

(vi) $c \bigodot \alpha \in V$, $\forall c \in \mathbb{F}, \alpha \in V$.

(vii) $c \bigodot (d \bigodot \alpha) = (c \cdot d) \bigodot \alpha$, $\forall c, d \in \mathbb{F}, \alpha \in V$.

(viii) $c \bigodot (\alpha \bigoplus \beta) = (c \bigodot \alpha) \bigoplus (c \bigodot \beta)$, $\forall c \in \mathbb{F}, \alpha, \beta \in V$.

(ix) $(c + d) \bigodot \alpha = (c \bigodot \alpha) \bigoplus (d \bigodot \alpha)$, $\forall c, d \in \mathbb{F}, \alpha \in V$.

(x) $1 \bigodot \alpha = \alpha$, where 1 denotes the multiplicative identity element.

Example 1. Let $V = \{(a_1, a_2, \ldots, a_n) : a_i \in \mathbb{R}, \forall i = 1, 2, \ldots, n\} = \mathbb{R}^n$ and $\mathbb{F} = \mathbb{R}$. The usual $\bigoplus$ is the vector addition, and the usual $+, \cdot$ are the the operations on field. Verify whether $(V, \bigoplus, \bigodot)$ is a vector space with respect to the field of scalars $\mathbb{R}$.
Solution. We start verifying the properties of the vector space stated earlier one-by-one. We choose α, β from V and find out that $\alpha + \beta \in V$ and $\alpha \bigoplus \beta = \beta \bigoplus \alpha$. Also, for $\gamma \in V$, $(\alpha \bigoplus \beta) \bigoplus \gamma = \alpha \bigoplus (\beta \bigoplus \gamma)$. Here we have the zero element $\theta = (0, 0, \ldots, 0) \in \mathbb{R}^n$ which assures that $\alpha \bigoplus \theta = \alpha$. Similarly, there does exist $-\alpha \in V$ for all α such that $\alpha + (-\alpha) = \theta$. In a similar fashion we deal with the rest of the properties, which is not so hard to check in this case.

Example 2. Let $V = \{A = (a_{ij}) : a_{ij} \in \mathbb{R}, 1 \leq i, j \leq 3\}$ be a set of all 3×3 matrices and $\mathbb{F} = \mathbb{R}$. With the usual matrix addition and scalar multiplication, verify that $(V, +, \cdot)$ is a vector space with respect to the field of scalars $\mathbb{R}$.
Solution. Do it by yourself.

Subspace

Let V be a vector space over a field of scalars $\mathbb{F}$ with respect to $\bigoplus$ and $\bigodot$, Let $W \subset V$. Then, W is said to be a subspace of V if W forms a vector space with respect to the same operations, i.e., $\bigoplus$ and $\bigodot$.

Example 1. Let $S = \left\{(x, y, z) \in \mathbb{R}^3 : y = z = 0\right\} \in \mathbb{R}^3$ and $\mathbb{F} = \mathbb{R}$. Prove that S is a subspace of $\mathbb{R}^3$.

Example 2. Let $S = \left\{(x, y, z) \in \mathbb{R}^3 : y > 0, z > 0\right\} \in \mathbb{R}^3$ and $\mathbb{F} = \mathbb{R}$. Prove that S is NOT a subspace of $\mathbb{R}^3$. (Hint: $\nexists\ \theta \in S$)

Example 3. Let $S = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2\right\} \in \mathbb{R}^3$ and $\mathbb{F} = \mathbb{R}$. Prove that S is NOT a subspace of $\mathbb{R}^3$. {Hint: Take $\alpha = (3, 45)$, $\beta = (-6, 8, 10) \in S$}