Advanced Engineering Mathematics
Lecture 26

Newton's Forward Interpolation

Let $x_0, x_1, \dots, x_n$ be equidistant and let $y_0 = f(x_0), y_1 = f(x_1), \dots, y_n = f(x_n)$. Also, suppose that $x_1 - x_0 = h, x_2 - x_1 = h, \dots, x_n - x_{n-1} = h$. Let $P_n(x)$ be a polynomial of degree n such that $y_i = f(x_i) = P_n(x_i), \forall i = 0, 1, \dots, n$. Further, let us assume that

$$P_n(x) = a_0 + a_1(x - x_0)^{(1)} + \dots + a_r(x - x_0)^{(r)} + \dots + +a_n(x - x_0)^{(n)}, \tag{1}$$

where $x^{(n)}$ is called the factorial polynomial and it is defined by

$$x^{(n)} = x(x-h)(x-2h)\dots(x - \overline{n-1}h).$$

In particular, $(x-x_0)^{(1)} = x - x_0$, $(x-x_0)^{(2)} = (x-x_0)(x - \overline{x_0 + h})$, $(x-x_0)^{(3)} = (x-x_0)(x - \overline{x_0 + h})(x - \overline{x_0 + 2h})$, etc.
The constants in (1) can be determined by

$$\begin{aligned} a_0 &= P_n(x), \\ a_1 &= \frac{P_n(x_1) - P_n(x_0)}{(x_1 - x_0)^{(1)}} - \left[a_2 \frac{(x_1 - x_0)^{(2)}}{(x_1 - x_0)^{(1)}} + \dots + a_n \frac{(x_1 - x_0)^{(n)}}{(x_1 - x_0)^{(1)}} \right], \\ a_r &= \frac{\Delta^r P_n(x)}{r! h^r} + \left[\text{term containing } \frac{(x_r - x_0) \text{ as the factors}}{r! h^r} \right]. \end{aligned} \tag{2}$$

Next, we set $x = x_0$ in the general formula (2) and get

$$a_r = \frac{\Delta^r P_n(x_0)}{r! h^r} = \frac{\Delta^r y_0}{r! h^r}. \tag{3}$$

Using (1) and (3), the required polynomial has the form:

$$P_n(x) = y_0 + \frac{(x - x_0)}{1! h} \Delta y_0 + \frac{(x-x_0)(x - \overline{x_0 + h})}{2! h^2} \Delta^2 y_0 + \dots.$$

Substituting u at the place of $\frac{(x-x_0)}{h}$, i.e., $x = x_0 + uh$ in the above formula to get

$$P_n(x) = P_n(x_0 + uh) = y_0 + \frac{u}{1!} \Delta y_0 + \dots + \frac{u(u-1)\dots(u - \overline{r-1})}{r!} \Delta^r y_0 + \dots. \tag{4}$$

Equation (4) is called the Newton's forward interpolation formula.

Example 1. Find the value of y at $x = 21$ from the below table.

Table 2.1				
Forward Difference Table				
x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
20	0.3420			
		0.0487		
23	0.3907		-0.0010	
		0.0477		-0.0003
26	0.4384		-0.0013	
		0.0464		
29	0.4845			

Solution. Using $u = \frac{x-x_0}{h} = \frac{21-20}{3} = \frac{1}{3}$ and the formula (4), we get

$$\begin{aligned} y = P_3(x) &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 \\ &= 0.3420 + \frac{1}{3}\cdot 0.0487 + \frac{1}{3}\frac{(-2)}{3}\frac{1}{2!}\cdot(-0.0010) + \frac{1}{3}\frac{(-2)}{3}\frac{(-5)}{3}\frac{1}{3!}\cdot(-0.0003) \\ &= 0.3583. \end{aligned}$$

Example 2. Find the value of $\sin 52°$ using the table 2.2.

Table 2.2				
Forward Difference Table				
x	$f(x) = \sin x$	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
45°	0.7071			
		0.0589		
50°	0.7660		-0.0057	
		0.0532		-0.0007
55°	0.8192		-0.0064	
		0.0468		
60°	0.8660			

Solution. Using the below formula

$$y = P_3(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

and $u = \frac{x-x_0}{h} = \frac{52°-45°}{5°} = 1.4°$, complete the solution.

Newton's Backward Interpolation

Following the same process as in previous subsection, the Newton's Backward Interpolation Formula has the below form, i.e., the interpolating polynomial is give by

$$P_n(x) = P_n(x_0 + vh) = y_0 + \frac{v}{1!}\nabla y_n + \ldots + \frac{v(v+1)\ldots(v+\overline{r-1})}{r!}\Delta^r y_0 + \ldots . \qquad (5)$$

where $v = \frac{x-x_0}{h}$

Example 1. The population in a town is given below:

Table 2.3					
Forward Difference Table					
x	$f(x) = \sin x$	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
1941	20				
		4			
1951	24		**1**		
		5		**1**	
1961	29		2		**0**
		7		1	
1971	36		3		
		10			
1981	46				

Then, find the population of town in 1946 and 1976.
Solution. Part I: We know $u = \frac{x-x_0}{h} = \frac{1946-1941}{10} = \frac{1}{2}$, and the formula

$$y = P_n(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$
$$\implies y(x = 1946) = P_n(1946) = \quad ?$$

Part II: We know $v = \frac{x-x_0}{h} = \frac{1976-1981}{10} = -\frac{1}{2}$ and therefore using table 2.4, we get

Table 2.4					
Forward Difference Table					
x	$f(x) = \sin x$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1941	20				
		4			
1951	24		1		
		5		1	
1961	29		2		**0**
		7		**1**	
1971	36		**3**		
		10			
1981	46				

$$y = P_n(x) = y_4 + v\nabla y_4 + \frac{v(v+1)}{2!}\nabla^2 y_4 + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_4 + \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_4$$
$$\implies y(x = 1976) = P_n(1976) = 40.5622.$$