Advanced Engineering Mathematics
Lecture 24

Newton-Raphson Method. Let α_0 be the approximate value of a root of $f(x) = 0$. Let α be the exact root. Then, $\alpha = \alpha_0 + h$, where h is very small.

$$f(\alpha) = 0 \Rightarrow f(\alpha_0 + h) = 0$$
$$\Rightarrow f(\alpha_0) + hf'(\alpha_0) + \frac{h^2}{2} f''(\alpha_0) + \cdots = 0$$

Since h is very small, we can neglect higher order of h.

$$\Rightarrow f(\alpha_0) + hf'(\alpha_0) = 0$$
$$\Rightarrow h = -\frac{f(\alpha_0)}{f'(\alpha_0)}$$

Therefore, $\alpha = \alpha_0 + h = \alpha_0 - \frac{f(\alpha_0)}{f'(\alpha_0)} \Rightarrow \alpha_n = \alpha_{n-1} - \frac{f(\alpha_{n-1})}{f'(\alpha_{n-1})}, \quad n = 1, 2, \cdots.$

Example 0.1. Find the root of $2x^3 - 8x - 6 = 0$ by Newton-Rapshon method.

Sol. $f(x) = 2x^3 - 8x - 6$. Now $f(1) = 2-3-6 = -7 < 0$. $f(2) = 2 \cdot 2^3 - 3 \cdot 2 - 6 = 4 > 0$. There exist a root in the interval $[1, 2]$. Let $x_0 = 2$. Also $f'(x) = 6x^2 - 3$. Now,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{4}{21} = 1.809524$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.784200$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.783769$$
$$x_4 = x_3 - \frac{f(x_1)}{f'(x_1)} = 1.783769$$

The required root is 1.783769 correct upto six decimal places.

Example 0.2. Find a positive root for $3x - \cos x - 1 = 0$.

Sol. Let $f(x) = 3x - \cos x - 1$. $f(0) = -2 < 0$ and $f(1) = 3 - \cos 1 - 1 \geq 1 > 0$. There is a root in $[0, 1]$. let $x_0 = 0.5$, therefore the Newton-Raphson scheme is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n \cos x_n - 1}{3 + \sin x_n}$$
$$\Rightarrow x_{n+1} = \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n}, \quad n = 0, 1, 2, \cdots.$$

System of linear equation.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Where, $A\vec{x} = \vec{b}$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\vec{x} = [x_1 \;\; x_2 \;\; x_3]^t, \qquad \vec{b} = [b_1 \;\; b_2 \;\; b_3]^t$$

1. Gauss-Seidal method. 2. Gauss- Jacobi method.

Gauss-Jacobi method.

$$\begin{aligned} a_1x_1 + b_1x_2 + c_2x_3 = d_1 \\ a_2x_1 + b_2x_2 + c_2x_3 = d_2 \\ a_3x_1 + b_3x_2 + c_3x_3 = d_3 \end{aligned}$$

Let us assume that, $|a_1| > |b_1| + |c_1|$, $|b_2| > |a_2| + |c_2|$, $|c_3| > |a_3| + |b_3|$.

The Jacobi method will converge if in each equation of the given system, the absolute value of the largest coefficient is greater than the sum of the absolute values of all the remaining coefficients. This is also called diagonally dominant.

The iteration scheme is given as follows,

$$\left.\begin{aligned} x^{(1)} &= \tfrac{1}{a_1}(d_1 - b_1y^{(0)} - c_1z^{(0)}) \\ y^{(1)} &= \tfrac{1}{b_2}(d_2 - a_2x^{(0)} - c_2z^{(0)}) \\ z^{(1)} &= \tfrac{1}{c_3}(d_3 - a_3x^{(0)} - b_3y^{(0)}) \end{aligned}\right\} \text{1st iteration}$$

$$\vdots$$

$$\left.\begin{aligned} x^{(n+1)} &= \tfrac{1}{a_1}(d_1 - b_1y^{(n)} - c_1z^{(n)}) \\ y^{(n+1)} &= \tfrac{1}{b_2}(d_2 - a_2x^{(n)} - c_2z^{(n)}) \\ z^{(n+1)} &= \tfrac{1}{c_3}(d_3 - a_3x^{(n)} - b_3y^{(n)}) \end{aligned}\right\} n+1 \text{ th iteration}$$

Gauss-seidal method. The iteration scheme is given as follows,

$$\left.\begin{aligned} x^{(1)} &= \tfrac{1}{a_1}(d_1 - b_1y^{(0)} - c_1z^{(0)}) \\ y^{(1)} &= \tfrac{1}{b_2}(d_2 - a_2x^{(1)} - c_2z^{(0)}) \\ z^{(1)} &= \tfrac{1}{c_3}(d_3 - a_3x^{(1)} - b_3y^{(1)}) \end{aligned}\right\} \text{1st iteration}$$

$$\vdots$$

$$\left.\begin{aligned} x^{(n+1)} &= \tfrac{1}{a_1}(d_1 - b_1y^{(n)} - c_1z^{(n)}) \\ y^{(n+1)} &= \tfrac{1}{b_2}(d_2 - a_2x^{(n+1)} - c_2z^{(n)}) \\ z^{(n+1)} &= \tfrac{1}{c_3}(d_3 - a_3x^{(n+1)} - b_3y^{(n+1)}) \end{aligned}\right\} n+1 \text{ th iteration}$$

Example 0.3. Solve via Gauss-Jacobi and Gauss-Seidal methods:

$$\begin{aligned} 10x - 5y - 2z &= 3 \\ 4x - 10y + 3z &= -3 \\ x + 6y + 10z &= -3 \end{aligned}$$

Sol. The coefficient matrix is, $\begin{bmatrix} 10 & -5 & -2 \\ 4 & -10 & 3 \\ 1 & 6 & 10 \end{bmatrix}$

For Gauss-Jacobi method,

$$\begin{aligned}
x^{(n+1)} &= \frac{1}{10}(3 - 5y^{(n)} - 2z^{(n)}) \\
y^{(n+1)} &= \frac{1}{10}(3 + 4x^{(n+1)} + 3z^{(n)}) \\
z^{(n+1)} &= \frac{1}{10}(-3 - x^{(n+1)} - 6y^{(n)})
\end{aligned}$$

Let us take, $\quad x^{(0)} = 0, \quad y^{(0)} = 0, \quad z^{(0)} = 0.$ Then,

$$\begin{aligned}
&x^{(1)} = 0.3 \quad y^{(1)} = 0.3 \quad z^{(1)} = -0.3 \\
&\dots \\
&x^{(8)} = 0.34 \quad y^{(8)} = 0.285 \quad z^{(8)} = -0.5051 \\
&x^{(9)} = 0.34 \quad y^{(9)} = 0.285 \quad z^{(9)} = -0.5053
\end{aligned}$$