Advanced Engineering Mathematics
Lecture 13

1 Integral Calculus

Leibnitz Rule of Differentiation. Let $f(x,y)$ and $f_x(x,y)$ are continuous on a rectangle $R=[a,b]\times[c,d]$. Let $\phi,\varphi:[a,b]\Rightarrow[c,d]$ be both differentiable on (a,b). Then the integral

$$g(x)=\int_{\phi(x)}^{\varphi(x)} f(x,y)dy$$

is differentiable on (a,b) and

$$g'(x)=\int_{\phi(x)}^{\varphi(x)} f_x(x,y)dy+f(x,\varphi(x))\varphi'(x)-f(x,\phi(x))\phi'(x)$$

Example 1.1. Evaluate by diff. under integral sign, the following integral $I=\int_0^1\frac{x^y-1}{\log x}dx$.

Soln. Let,

$$\begin{aligned}
\phi(y) &= \int_0^1 \frac{x^y-1}{\log x}dx. \qquad \text{(Using Leibnitz rule of derivative)}\\
\Rightarrow \phi'(y) &= \int_0^1 \frac{\partial}{\partial y}\Big(\frac{x^y-1}{\log x}\Big)\,dx=\int_0^1\frac{1}{\log x}\frac{\partial}{\partial y}(x^y-1)\,dx=\int_0^1\frac{1}{\log x}x^y\log x\,dx\\
\Rightarrow \phi'(y) &= \int_0^1 x^y\,dx=\Big[\frac{x^{y+1}}{y+1}\Big]_0^1=\frac{1}{y+1}\\
\Rightarrow \phi(y) &= \int\frac{1}{y+1}\,dy=\log_e y+c
\end{aligned}$$

Again we know that,

$$\phi(y)=\int_0^1\frac{x^y-1}{\log x}\,dx\Rightarrow\ \phi(0)=\int_0^1 0\,dx=0$$
$$\phi(0)=\log_e 0+1+c\Rightarrow\ 0=0+c\Rightarrow\ c=0$$

The required value of the integral is $\phi(y)=\log(1+y)$.

Example 1.2. Evaluate $I=\int_0^\infty e^{-x^2}\cos\alpha x\,dx$.

Soln. Let,

$$\begin{aligned}
I(\alpha) &= \int_0^\infty e^{-x^2}\cos\alpha x\,dx\\
\Rightarrow\ \frac{dI}{d\alpha} &= -\int_0^\infty xe^{-x^2}\sin\alpha x\,dx\\
&= \Big[\frac{1}{2}e^{-x^2}\sin\alpha x\Big]_0^\infty-\frac{\alpha}{2}\int_0^\infty e^{-x^2}\cos\alpha x\,dx \qquad \text{(Integrating by parts)}\\
&= \frac{\alpha}{2}I(\alpha)\\
\Rightarrow\ \frac{d\alpha}{I} &= -\frac{\alpha}{2}d\alpha\Rightarrow\ \log I(\alpha)=-\frac{\alpha^2}{4}+c\\
&\Rightarrow\ I(\alpha)=ce^{-\frac{\alpha^2}{4}} \qquad (1.1)
\end{aligned}$$

Since, $I(\alpha) = \int_0^\infty e^{-x^2} \cos \alpha x \, dx \Rightarrow I(0) = \int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}$.
Now, from (1.1) $\frac{\sqrt{\pi}}{2} = I(0) = ce^0 = c \Rightarrow c = \frac{\sqrt{\pi}}{2}$. Therefore, the required value of the integral is $I(\alpha) = \frac{\sqrt{\pi}}{2} e^{-\frac{\pi^2}{4}}$.

Integration in $\mathbb{R}^2$ (double integral) and in $\mathbb{R}^3$(triple integral).

$$f : [a,b] \Rightarrow \mathbb{R}, \qquad I = \int_a^b f(x)\, dx$$
$$f : [a,b] \times [c,d] \Rightarrow \mathbb{R}, \qquad I_1 = \int_a^b \int_c^d f(x,y)\, dx\, dy$$
$$f : [a,b] \times [c,d] \times [e,f] \Rightarrow \mathbb{R}, \qquad I_1 = \int_a^b \int_c^d \int_e^f f(x,y,z)\, dx\, dy\, dz$$

Example 1.3. Evaluate $I = \int_c \frac{dx}{x+y}$, where c is the curve $x = at^2, y = 2at, 0 \leq t \leq 2$.

Soln. We have, $x = at^2, y = 2at \Rightarrow dx = 2atdt, dy = 2adt$. Now,

$$\begin{aligned} I &= \int_0^2 \frac{2atdt}{at^2 + 2at} \\ &= 2\int_0^2 \frac{dt}{t+2} \\ &= 2\big[\log(t+2)\big]_0^2 = 2[\log 4 - \log 2] = 2\log 2 \end{aligned}$$