Advanced Engineering Mathematics
Lecture 12

1 Integral Calculus

3. $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, $m > 0,\ \ n > 0$.

4. $\sqrt{\pi}\Gamma(2m) = 2^{2m-1}\Gamma(m)\Gamma(m+\frac{1}{2}),\ \ m > 0 \Rightarrow$ Legendre's Duplication Formula.

5. $B(m,n) = 2^{1-2m}B(m,\frac{1}{2})$

Example 1.1. Find the value of $\Gamma(\frac{1}{2})$.

Soln. Let $m = n = \frac{1}{2}$, then using properties of Beta and Gamma functions, we get

$$B(\frac{1}{2},\frac{1}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})} \Rightarrow [\Gamma(\frac{1}{2})]^2 = B(\frac{1}{2},\frac{1}{2})$$

Again,

$$\begin{aligned} B(\frac{1}{2},\frac{1}{2}) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}dx &= \int_0^1 \frac{dx}{\sqrt{x}\sqrt{1-x}} \quad [x = \sin^2(\theta) \Rightarrow dx = 2\sin\theta\cos\theta d\theta] \\ &= 2[\theta]_0^{\frac{\pi}{2}} \\ &= \pi \end{aligned}$$

Hence, $\Gamma(\frac{1}{2}) = \sqrt{B(1,1)} = \sqrt{\pi}$

Example 1.2. Evaluate $\displaystyle\int_0^\infty e^{x^2}dx$.

Soln. Let $x^2 = z \Rightarrow 2xdx = dz$. So we get,

$$\begin{aligned} I &= \int_0^\infty e^{x^2}dx \\ &= \frac{1}{2}\int_0^\infty e^{-z}z^{-\frac{1}{2}}dz \\ &= \frac{1}{2}\int_0^\infty e^{-z}z^{\frac{1}{2}-1}dz \\ &= \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2} \end{aligned}$$

Example 1.3. Evaluate $I = \displaystyle\int_0^1 \frac{xdx}{\sqrt{1-x^5}}$

Soln. Let us take,

$$\begin{aligned} x^5 = z \ &\Rightarrow 5x^4dx = dz \\ &\Rightarrow dx = \frac{dz}{5x^4} \\ &\Rightarrow dx = \frac{1}{5}z^{-\frac{4}{5}}dz \end{aligned}$$

Now,

$$\begin{aligned}
I &= \int_0^1 z^{\frac{1}{5}} \frac{1}{5} z^{-\frac{4}{5}} (1-z)^{-\frac{1}{2}} dz \\
&= \frac{1}{5} \int_0^1 z^{-\frac{3}{5}} (1-z)^{-\frac{1}{2}} dz \\
&= \frac{1}{5} \int_0^1 z^{\frac{2}{5}-1} (1-z)^{\frac{1}{2}-1} dz \\
&= B(\frac{2}{5}, \frac{1}{2})
\end{aligned}$$

Example 1.4. Evaluate $I = \displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^4}}$ in terms of Gamma function.

Soln. Let us take,

$$\begin{aligned}
x^4 = z \ &\Rightarrow 4x^3 dx = dz \\
&\Rightarrow dx = \frac{dz}{4x^3} \\
&\Rightarrow dx = \frac{1}{4} z^{-\frac{3}{4}} dz
\end{aligned}$$

Now,

$$\begin{aligned}
I &= \int_0^1 \frac{1}{4} z^{-\frac{3}{4}} dz \frac{1}{\sqrt{1-z}} \\
&= \frac{1}{4} \int_0^1 z^{\frac{1}{4}-1} (1-z)^{\frac{1}{2}-1} dz \\
&= \frac{1}{4} B(\frac{1}{4}, \frac{1}{2}) \\
&= \frac{1}{4} \frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4}+\frac{1}{2})} = \frac{1}{4} \frac{\sqrt{\pi}\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} = \frac{\sqrt{\pi}}{4} \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} = \frac{1}{8}\sqrt{\frac{2}{\pi}} [\Gamma(\frac{1}{4})]^2
\end{aligned}$$

$[\because \Gamma(\frac{3}{4})\Gamma(\frac{1}{4}) = \sqrt{2}\pi]$

Differentiation under integral sign. Suppose f is defined on a rectangle $R : [a, b] \times [c, d]$ and it is continuous on R. Let $f_x(x, y)$ exists and continuous on R, then the integral

$$g(x) = \int_c^d f(x, y) dy$$

is differentiable with respect to x on (a, b) and

$$g'(x) = \int_c^d f_x(x, y) dy$$